A Conceptual Framework Based on Fractal Geometry for Design, Modeling and Rapid Prototyping of Complex Geometric Shapes

Dina Rochman[1] and Sergio Vázquez[2]

[1]Universidad Autónoma Metropolitana, Cuajimalpa drochman@correo.cua.uam.mx
[2]Universidad Autónoma Metropolitana, Cuajimalpa smonterrosas@correo.cua.uam.mx

ABSTRACT

In the "Project GEMA" geometry and mathematics, 62 prototypes were created to establish a conceptual and formal language of the internal architecture of the solids that maintains a progressive continuity through mathematics, to explore and analyze the changes generated by the points, lines, planes and volume of the cube, the sphere and the tetrahedron. The idea of this project came from mixture of geometric graphics and mathematics to construct models in three dimensions using CAD applications and rapid prototyping through the resources and elements of descriptive geometry. This paper illustrates step by step the construction of the "Project GEMA"® cube # 022, from the standpoint of mathematics and design with CAD applications.

Keywords: spatial reasoning, polyhedra, digital technology, rapid prototyping.
DOI: 10.3722/cadaps.2013.307-319

1 INTRODUCTION

The "Project GEMA" geometry and mathematics [4], focuses on the perception of the three dimensions through modeling and rapid prototyping of geometric shapes, to observe and understand the compositions of a mathematical pattern that expresses harmony, scale and proportion [2].

The aim of this project is investigate, experiment, explore and evaluate the design and modeling process of complex shapes to conceptualize three dimensions, and even to create a method of spatial reasoning, to understand and interpret the different elements of descriptive geometry and perform the necessary operations to create rapid prototypes.

In this project, CAD applications are used as part of the procedures and techniques currently used for rapid prototyping, based on the following methodology:

a) Knowledge, in this first stage, geometric shapes are traced, using the resources and elements that gives us descriptive geometry.

b) Evaluation, in this second stage, each of the shapes is evaluated from the standpoint of functional, aesthetic, constructive and technological, to analyze their advantages and disadvantages for rapid prototyping.

c) Development, in this third stage involves modeling each of the shapes.

d) Analysis, in this fourth stage, to get to know the properties of the shapes, it is necessary to create an animation to see the figure from all points of view, since only in this way, it can see the shapes from the inside out and vice versa.

e) Execution, in this fifth stage, the file is saved in STL, 3DS or VRML format and passed to the 3D printer for rapid prototyping.

f) Completed, in this last stage is the cleaning and infiltration of the shape.

This process creates a new concept of learning in the creation of rapid prototyping, which fulfills a very important role, the development of the spatial factor, what is known as the ability to "see the space," the translation of the mental representation of the object from two to three dimensions, through the knowledge and skills acquired to solve a given problem by:

1. Geometrical language: it is a mechanical process of learning the resources and elements of orthographic projections, and then, through practice, learning how to use them correctly.

2. Conceptualization of the three-dimensionality: when we understands the elements in space through the resources from Descriptive Geometry.

3. Development of a method of spatial reasoning: when we think carefully to perceive the image of an object through the mental representation of it, and then we define the objectives and the limits of the method, we analyze situations, we choose the elements to synthesize and, finally, we find the best solutions.

This project consists of 62 prototypes, using the properties of: the cube, the sphere and the tetrahedron, and for its development took into account the mathematical principles of Cantor, Archimedes, and Thales of Miletus theorems.

Fig 1: Prototypes: (a) Cube, (b) Sphere, (c) Tetrahedron.

It is impossible to present all the work done in so few pages, so this paper is organized as follows: in section 2 gives a brief explanation of Descriptive Geometry, in Section 3, we explain the mathematical principles that were used to create the cubes, the spheres and the tetrahedron. In Section 4, it is illustrates step by step the construction of the cube module # 022 from the standpoint of mathematics and design using CAD applications. Section 5 shows the final cube # 022 that consists of 20 modules. Section 6 presents the results and finally Section 7, the conclusions. We want to mention that all the

figures presented in this paper are original and created by the authors at the Autonomous Metropolitan University, Cuajimalpa in Mexico City.

2   DESCRIPTIVE GEOMETRY

As we know, the geometry is a creation of Greek mind, its origins date back to earlier times in Mesopotamia and Egypt, and it has its uses descriptive when Gaspar Monge, (1746-1818 French military engineer), made a graphical analysis of the representation of points in space (dihedral system), and gave rise to science we know today as descriptive geometry (1799), Ïpart of mathematics that represent the projections of plane figures in spaceÏ.

When we speak of space, we talk about ÎdistanceÏ, [1] and these are represented in orthogonal projections, and to produce geometric shapes, we do through the elements of geometry: point, line, plane and volume. But today we can say that has changed the way to create geometric shapes, due to the use of the innovative digital technologies, i.e., now we are modeling geometric shapes, where it is necessary understand the relationship between geometry and virtual space where a mathematical language is used to transform, through mathematical patterns, complex geometric shapes that years ago were impossible to do, and today are generated in CAD applications and rapid prototyping.

In geometry, the geometric shapes are classified in polyhedral and round shapes. We define a polyhedron as a solid figure whose faces are polygonal and are classified as prisms and pyramids. A prism is a solid formed by two polygons parallel to each other at their bases and when we define its height parallelograms are created with the number of faces that have the base. The prisms that we know are the cube, and based prisms of three, four, five, six, or N-sides.

The cube or regular hexahedron is a polyhedron of six congruent square faces, and is one of the Platonic solids. We can also classify the cube as a straight, rectangular parallelepiped, because their faces are four sides in parallel pairs. The regular hexahedron, satisfies the Euler polyhedron theorem, ($c + v = a + 2$, where "c" is the number of faces, "v" is the number of vertices and "a" is the number of edges), because it has six faces, eight vertices and twelve edges, $8 + 6 = 12 + 2$.

A pyramid in geometry is a solid formed by a polygon in its base, and when we define the vertex isosceles triangular planes are created in the number of faces that have the base. The pyramids we know are: the tetrahedron, and based pyramids of three, four, five, six, or N-sides.

A tetrahedron is a polyhedron formed by four equilateral triangles, and four vertices. It is one of five perfect polyhedral called Platonic solids, and meets the Euler polyhedron theorem, $4 + 4 = 6 + 2$.

The round shapes are solids that have curved surfaces and are classified as cylinder, cone and solids of revolution as the ellipsoid, paraboloid, hyperboloid, torus and sphere. A sphere in geometry is a solid body bounded by a curved surface where all points are equidistant from an inner called center of the sphere. The sphere, as a solid of revolution is generated by rotating a semi-circular area around its diameter (Euclid, L XI). But we can also build a sphere with pyramids (Rochman, 1996). The pyramid that we use to construct a sphere is formed at its base by four-sides whose vertex is the center of the sphere and formed in its sides equal isosceles triangles and their edges measure the same thing.

Fig 2: Prototype: (a) Cube, (b) Tetrahedron, (c) Sphere.

3   MATHEMATICAL PRINCIPLES

The mathematician George Cantor was able to establish a correspondence between points discontinuous in space "p" dimensional and the interval [0, 1] called the Cantor set. In Cantor´s theorem it divide the interval [0, 1] into three equal intervals, it is removed the central interval, and kept the closed intervals of the first level. If Cantor´s set is traced on a two-dimensional graphical representation the result is a 3x3 grid base, known as fractal [3].
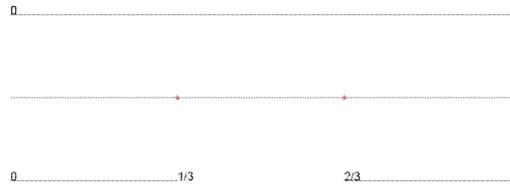
Fig 3: Cantor' theorem: Interval [0, 1] first level.
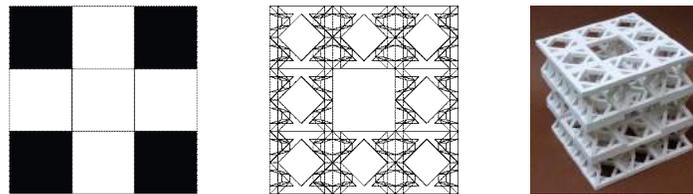
Fig 4: Cantor' theorem: (a) Grid 3x3, (b) Orthogonal projection cube, (c) Prototype cube.

Archimedes, in his work in Siracuza in the year 287 BC, tells us that if we have a circle of radius 1, its area is exactly equal μ $(1)^2 = \mu$. To approximate the number μ, Archimedes devised a regular octagon inscribed in a circle, the area found was 2.8, i.e., 2.8 < μ, doubling the number of sides of the circle, i.e., using 16-sides regular polygons inscribed in the circumference, we get areas much closer to μ. So on, we can build polygons inscribed with large numbers of sides, which means that their fields are increasingly closer to the area of the circle, which is μ.
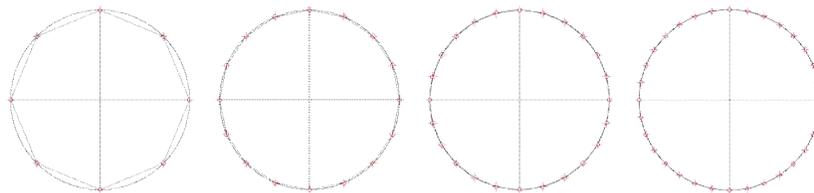
Fig 5: Archimedes' theorem: (a) 8_sides polygon, (b) 16_sides polygon, (c) 24_sides polygon, (d) 32_sides polygon.
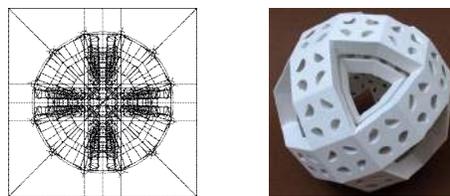
Fig 6: (a) Orthogonal projection sphere, (b) Prototype sphere.

The Thales of Miletus theorem says that if three lines, a, b, c, cut two secant lines r, r ', the resulting segments are proportional.
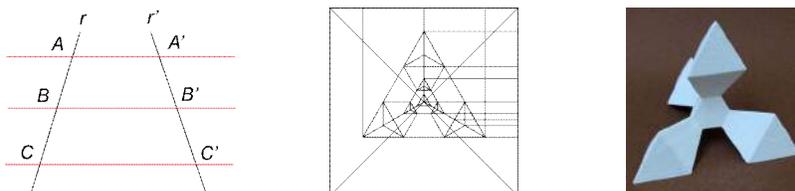
Fig 7: (a) Miletus´ theorem: Proportional segments, (b) Orthogonal projection tetrahedron, (b) Prototype tetrahedron.

4   CONSTRUCTION OF MODULE

This section will begin by presenting the render, prototype and orthogonal representations in two and three-dimensional "Project GEMA"® [5].cube module # 022, where it can be seen the geometrical shape to understand the steps that below are illustrated.
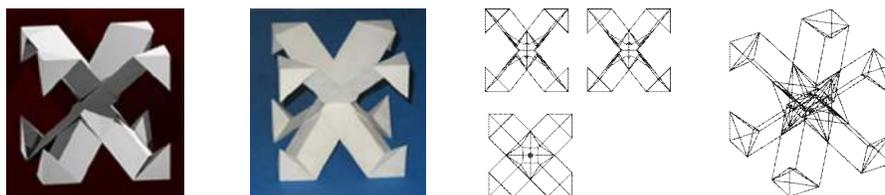
Fig 8: (a) Render, (b) Prototype, (c) Orthogonal projection, (d) Three-dimensional graphical representation.

4.1 STEP BY STEP CONSTRUCTION "PROJECT GEMA"® CUBE MODULE # 022

4.1.1 Step one

The basis of our design is a cube that measures 300 x 300 x 300 mm, so in this first step we located a center from the origin.
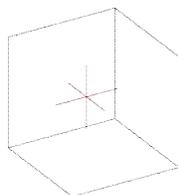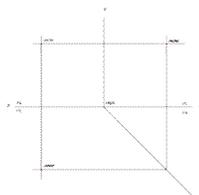
Origin		Center	
X	0	X	350
Y	0	Y	350
Z	0	Z	350

Fig 9: Center of the cube module.

4.1.2 Step two

In the second step we trace and model four cubes, the measure of each one is 300 x 300 x 300 mm, where the cube_one remains in its original position, the cube_two is rotated 45° on the axis "z" in the vertical plane, the cube_three is rotated 45° on the axis "y" in the horizontal plane, and the cube_four is rotated 45° on the axis ÎxÏ in the lateral plane.
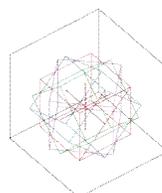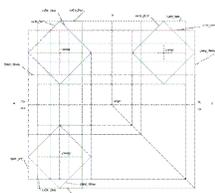
Fig 10: The four cubes in space.

4.1.3 Step three

Before proceeding to find the intersections in the orthogonal projections, in this step we will analyze mathematically the distances of intersection between the cube_one with cube_two, to give the numerical values. This same process is done with the other cubes, due the length of the work is not possible to write all data.
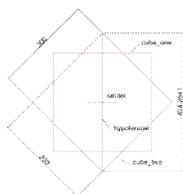
We find the measure of the hypotenuse of cube_two:

$$d = L \sqrt{2}$$
$$d = 300 \times 1.4142$$
$$d = 424.2641$$
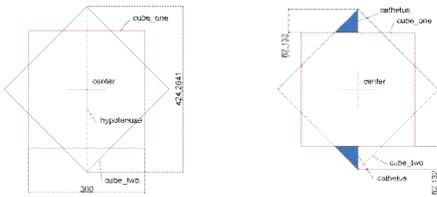
We find the measure of each of the cathetus of the triangles protruding from cube_one, as follows, we subtract from the measure of the hypotenuse of cube_two the measure of the side of cube_one, and divided between two.

$$a = (424.26 Ì 300) / 2$$

$$a = 124.2641 / 2$$

$$a = 62.132$$
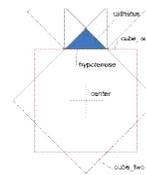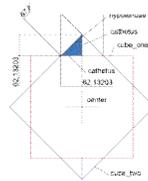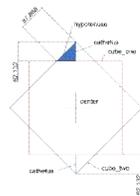
We find the measure of the hypotenuse of the triangle protruding from cube_one.

$$d = L \text{ ¿} 2$$

$$d = 62.13 \times 1.4142$$

$$d = 87.868$$

We find the measure of the hypotenuse of the triangle. We can do it in two ways: by multiplying the distance of the cathetus by two, or finding the measure of the hypotenuse.

$$c = L * 2$$

$$c = 62.13203436 * 2$$

$$c = 124.2640687$$

$$d = L \text{ ¿} 2$$

$$d = 87.86796565 \text{ ¿} 2$$

$$d = 87.867965665 * 1.4142$$
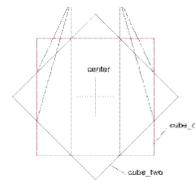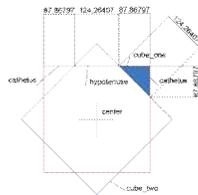
$$d = 124.2640687$$

We find the measure of the cathetus of the triangle protruding from cube_two.

$$a = (300 Ì 124.2640687) / 2$$

$$a = 87.86796565$$

We find the intersections points between cube_one and cube_two. As we can see we have eight intersections between the two cubes.

Cube_one				Cube_two			
vertex	x	y	z	vertex	x	y	z
1	500	500	200	9	562	350	200
2	200	500	200	10	350	562	200
3	200	500	500	11	350	565	500
4	500	500	500	12	562	350	500
5	500	200	200	13	350	138	200
6	200	200	200	14	138	350	200
7	200	200	500	15	138	350	500
8	500	200	500	16	350	138	500

Tab 1: Location of the vertices of cube_one and cube_two, from the origin (0,0,0).

Points of intersection cube_one with cube_two							
vertex	x	y	z	vertex	y	z	x
17	412.1320344	500	200	25	500	287.867966	200
18	287.8679656	500	200	26	200	287.867966	200
19	287.867966	500	500	27	200	287.867966	500
20	412.132034	500	500	28	500	287.867966	500
21	500	412.132034	200	29	412.132034	200	200
22	200	412.132034	200	30	287.867966	200	200
23	200	412.132034	500	31	287.867966	200	500
24	500	412.132034	500	32	412.132034	200	500

Tab 2: Location of the points of intersection of cube_one and cube_two, from the origin (0,0,0).

4.1.4 Step four

In the fourth step we find orthographically the intersections of: the cube_one with cube_two, the cube_two with cube_three and the cube_three with cube_four. Once we identified the intersections, the geometric structures are truncated so as to obtain eight irregular geometric shapes, in which each are located at the vertexes of the base cube. As we see the intersections of the four cubes was performed in a counter-clockwise, because it is necessary that the basis of irregular shapes are directed into the center of the base cube.
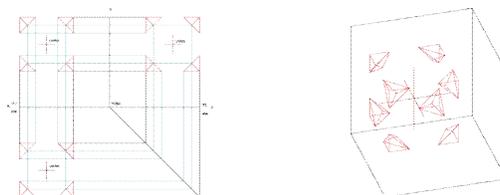
Fig 11: The eight irregular geometric shapes.

4.1.5 Step five

In the fifth step we trace and model inclined prisms, ranging from the base of each of the geometric structures toward the cube center, where the eight prisms intersect to form a single shape.
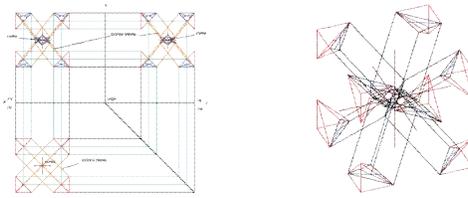
Fig 12: The eight irregular geometric shapes with inclined prism.

4.1.6 Step six

In this last step saves the file in STL format to send to the 3D printer.
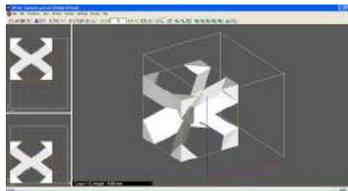
Fig 13: STL file.

5   "PROJECT GEMA"® CUBE # 022

To generate the final cube, 20 modules are joined, considering Cantor's theorem. There is no rule to accommodate each module; we can rotate, to create different visual effects, each module in the "x", "y", "z" axis with an angle of 90°.
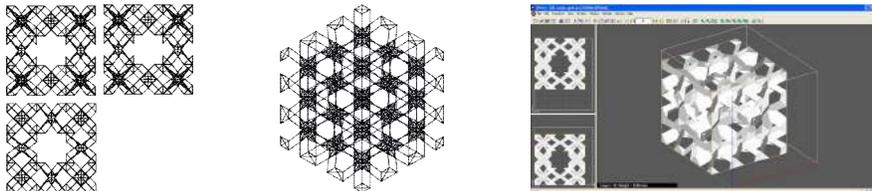
Fig 14: (a) Orthogonal projection, (b) Three–dimensional graphical representation (c) STL file.
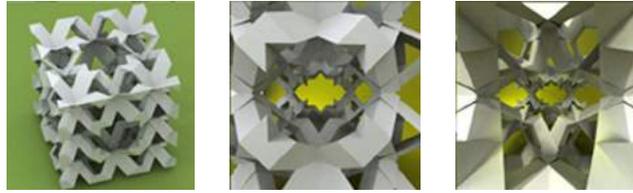
Fig 15: Render from various points of view.

Fig 16: Prototype from various points of view.

6   RESULTS

The results obtained in the design and modeling of the "Project GEMA"® cube # 022 based on fractal geometry, led us to create new designs using Cad applications, since in this process was built a rapid prototyping dimensionally stable to support its own weight, was obtained the transmission of a visual language through modeling structurally balanced, and was established a conceptual language for creating complex shapes. The use of CAD applications in this project allowed us to model, analyze and visualize the mathematical patterns of the fractal geometry of each of the models created, as we can see below in the Fig. 17, Fig. 19, Fig 21, Fig. 23 and Fig. 25 the modules that were used to generate the new models, then we note in the Fig. 18, Fig. 20, Fig. 22, Fig. 24 and Fig. 26 that although the arrangement of the twenty modules is very different in each of the models, each model meets the characteristics of fractal geometry [3]: 1) look the same at any scale of observation 2) have length infinite, 3) not differentiable and 4) have fractional or fractal dimension.
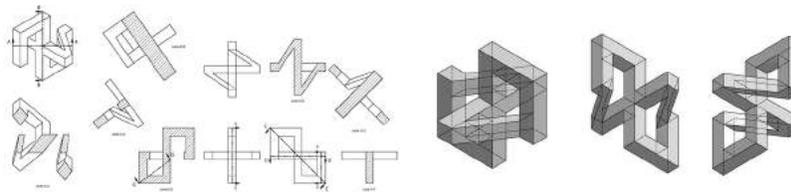
Fig 17: (a) Orthogonal projection, (b) Three-dimensional graphical representation.
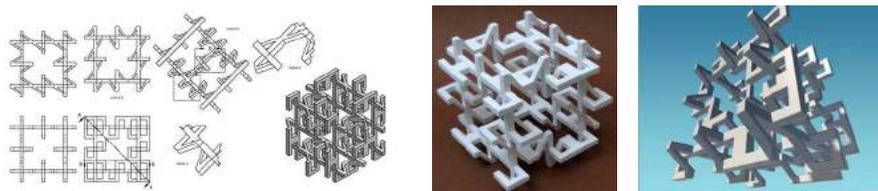
Fig 18: (a) Orthogonal projection, (b) Prototype, (c) Rendered image.
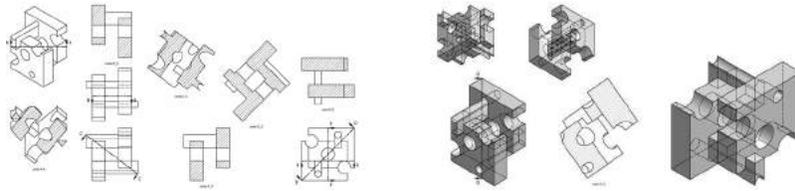
Fig 19: (a) Orthogonal projection, (b) Three-dimensional graphical representation.
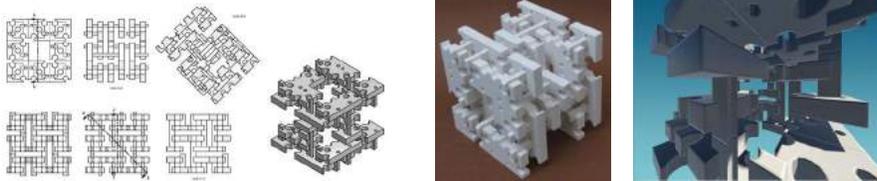
Fig 20: (a) Orthogonal projection, (b) Prototype, (c) Rendering.
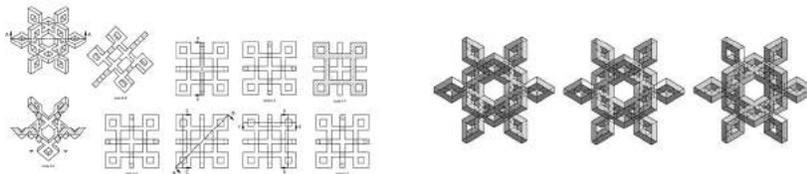
Fig 21: (a) Orthogonal projection, (b) Three-dimensional graphical representation.
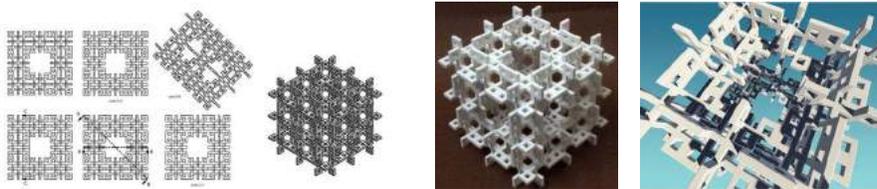
Fig 22: (a) Orthogonal projection, (b) Prototype, (c) Render.
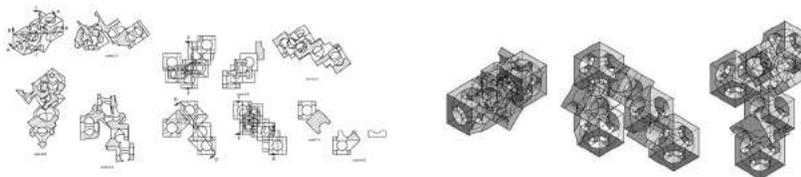
Fig 23: (a) Orthogonal projection, (b) Three-dimensional graphical representation.
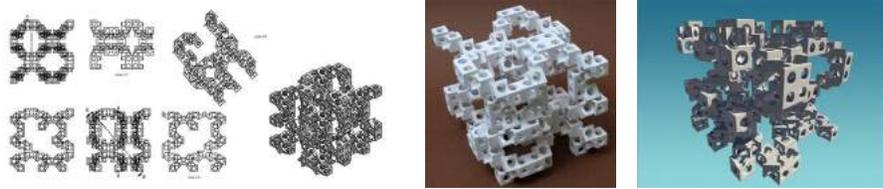
Fig 24: (a) Orthogonal projection, (b) Prototype, (c) Rendering.
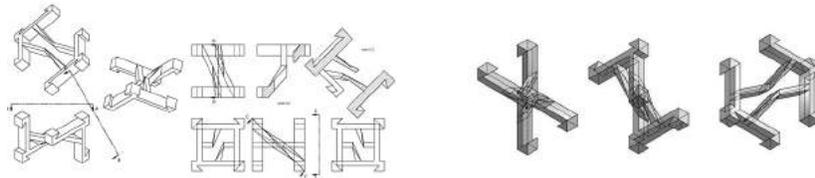
Fig 25: (a) Orthogonal projection, (b) Three-dimensional graphical representation.
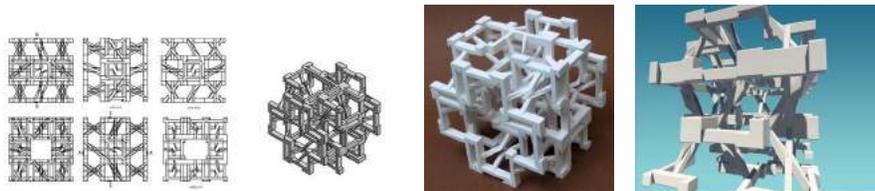
Fig 26: (a) Orthogonal projection, (b) Prototype, (c) Render.

7   CONCLUSIONS

As we can see in this project, the activity of designing prototypes is linked to creative thinking and the ability to understand three-dimensional space, where the geometric graphics and mathematics, are the vehicle that encourage self-expression, imagination, creativity and spatial development to understand the characteristics of orthogonal projections in two and three dimensions and their interrelationships.

8   ACKNOLEDGEMENTS

9   REFERENCES

[1]   De la Torre, C. Miguel.: Geometría descriptiva, UNAM, México, 1975.
[2]   Kimberly, E.: Geometría del diseño, estudio en proporción y composición, Editorial Trillas, México, 2003.

[3] Mandelbrot, B.: La geometría fractal de la naturaleza, Oceano–Tusquets Editores, Barcelona, 1997.

[4] Rochman, D.: Proyecto GEMA, (geometría matemática), Proceso de diseño y modelado del cubo, la esfera y el tetraedro a la realización del prototipo, Proyecto de investigación, Universidad Autónoma Metropolitana, Cuajimalpa. México, 2010.

[5] ®SEP-INDAUTOR 03–2011–062911013000–14 Título: PROYECTO GEMA (geometría matemática) ÎProceso de diseño y modelado del cubo, la esfera y el tetraedro a la realización del prototipoÏ, Tipo: registro de colección de obra, Ciudad de México.