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Transition Curve Modeling with Kinematical Properties: Research on Log-Aesthetic Curves

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ABSTRACT

Log-aesthetic curves (LACs), which are generally used in industrial design for aesthetic shape modeling, are examined for implementation as transition curves despite their non-trivial representation in terms of incomplete gamma functions, or Fresnel integrals which appear in the parametric equations of a certain spirals. The family of log-aesthetic curves includes well-known spirals as Euler, Nielsen, logarithmic spiral, and involutes of a circle. The horizontal geometry of the route can contain classical transition curves, which are formed by transition curve-circular arc-transition curves. In order to compare the examined family of log-aesthetic transition curves with the classical transition curve in terms of vehicle-road kinematics, the curvature and superelevation functions are derived, and functions of lateral change of acceleration (LCA) curves are obtained and illustrated graphically using a constant motion model. The discontinuities in the form of jumps in the graphs of the lateral change of acceleration are taken into consideration in order to compare log-aesthetic curves and clothoids.

Keywords: log-aesthetic curve, pseudospiral, lateral change of acceleration, curvature, route alignment, monotone curvature, spiral.

1. INTRODUCTION

Mathematical design is an integrative scientific and artistic direction, the toolkit of which aims to establish a process for the artistic design of the environment and its components on the basis of the mathematical analysis of problems, in order to obtain a product of design whose aesthetic and technical properties are optimized with the help of precise calculations [20].

According to this definition, so-called *log-aesthetic curves* (LACs) can be considered to be elements of mathematical design, since they have aesthetic appeal [9,10,27] and are represented in terms of incomplete gamma functions [32]. Additionally, their plastic properties have been analyzed in terms of technical aesthetics laws [36,37], and their possible applications such as, for example, gear design (Fig. 1), have been discussed in [35].

There are many interesting manuscripts on LACs. The general formula of LACs, which describes the relationship between their radii of curvature and their lengths, has been studied by [21][1] and [13–15]. [16] have proposed a method to generate a LAC with G^2 continuity from a sequence of 2D points. Discrete log-aesthetic filters to fair a sequence of points with noises and to fit them locally to LACs were proposed in [17]. [18] have reformulated the LAC with a variational principle and have proposed several new

[1] In his book they were known as *pseudospirals*.

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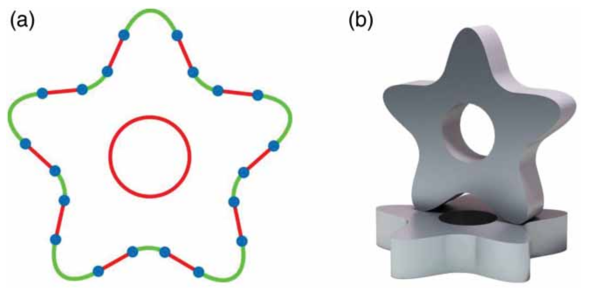

Fig. 1: Application of a G^2 multi-spiral transition curve in gear design. The blue points in (a) denote the junctions of LAC segments (green) with straight line segments (red) (for the interpretation of the references in order to color in this figure legend, the reader is referred to the web version of this article)

functionals to be minimized for free-form surfaces and defining the log-aesthetic surface. In their recent work, [19] have proposed a novel method to solve the G^2 Hermite interpolation problem with LACs, which takes the form of log-aesthetic triplets and is implemented as a plug-in module for a commercial CAD system as well.

There are a series of serious and interesting works by Norimasa Yoshida and his collaborators. For example, in [28] the authors proposed quasi-aesthetic curves that can be used in CAD systems for aesthetic shape design; in [27], a novel method for drawing an aesthetic curve segment by specifying two endpoints and their tangent vectors has been proposed; in [29], a method for the interactive generation of compound-rhythm log-aesthetic space curve segments was discussed.

Despite the above-mentioned works, LACs have never been studied in terms of road kinematics, and this would be the aim of our current research work.

Motor vehicles may not follow their normal lane widths when they meet a circular horizontal curve. The resultant lateral forces, which occur at the connection points of alignment and circular curves, cannot be eliminated immediately. So that vehicles can follow their lanes with a longer path, this is satisfied by adding one more curve having different radii [1].

Transition curves are extremely important and are widely used in the design of today's route alignment. They are used for joining straight lines with a circle or a circle of radius R_1 with a circle of radius R_2. The *Euler spiral* (*clothoid*) is the most commonly used transition curve, which can respond to demands of up to 120 km/h speeds in route alignment. The inadequacy of existing curves has been identified at high speeds, e.g., 200 km/h – 500 km/h, in railways. Curves such as the *Bloss curve, sinusoid, cosinusoid, Tari 1* [23] and *Tari 2* [23] have better specifications with regard to vehicle-road dynamics [24]. The discontinuities in the form of jumps in the graphs of the lateral change of acceleration function for these curves are eliminated. Important components of vehicle-road dynamics are:

- Vehicle speed, acceleration, velocity models;
- Technical and physical specifications of vehicles, weather conditions;
- Horizontal-vertical geometry of route;
- Superelevation, superelevation functions;
- Transition curve usage;
- Lateral change of acceleration functions [3].

1.1. Lateral Change of Acceleration

Lateral change of acceleration is the change of the resultant acceleration introduced along the curve normal with respect to the time T (sec). It can be expressed as:

$$z = \frac{d\vec{a}}{dT}\vec{n}, \tag{1}$$

where $\vec{a}$ is the resultant acceleration formed by all free forces and tangential acceleration. Fig. 2 illustrates the forces acting on a vehicle moving on a superelevated road: the gravitational force (P=mg), the centrifugal force (F=mkv^2) and the motor force (F_T =ma_T). The resultant acceleration is expressed as:

$$\vec{a} = \frac{dv}{dt}\vec{t} + \frac{b}{\sqrt{u^2+b^2}}\left(kv^2 - g\frac{u}{b}\right)\vec{n}, \tag{2}$$

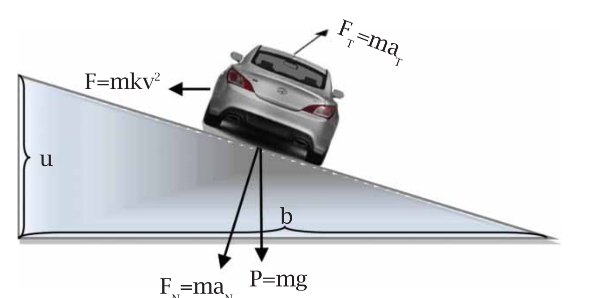

Fig. 2: Cross section of the forces acting on a vehicle moving on a superelevated road.

where k – the curvature of the orbiting curve defined on the horizontal plane (1/m); g – the gravitational acceleration (9.81 m/sec^2); b – the horizontal width of the road platform (m); m – mass (kg); u – superelevation (m); v – the velocity (m/sec); $\vec{t}$ is the unit tangent vector. The equation of lateral change of acceleration is derived from Eqs. (1) and (2):

$$z = \frac{d\vec{a}}{dT}\vec{n} = \frac{bv}{\sqrt{u^2+b^2}}\left(3ka_t + v^2\frac{dk}{dl} - \frac{kv^2u+gb}{u^2+b^2}\frac{du}{dl}\right) \tag{3}$$

where a_t = the tangential acceleration produced by the motor (engine) force (m/sec^2).

Some references give the maximum magnitude of lateral change of acceleration as:

- in [22], $Z_{MAX} = 0.3\,\mathrm{m/sec}^3$ for highways;
- in [4], $Z_{MAX} = 0.3\,\mathrm{m/sec}^3$ for railways;
- in [7], $Z_{MAX} = 0.4\,\mathrm{m/sec}^3$ for railways;
- in [26], $Z_{MAX} = 0.6\,\mathrm{m/sec}^3$ for highways;
- in [25], $Z_{MAX} = 0.6\,\mathrm{m/sec}^3$.

Three elements in Eq. (3): $k = k(l)$ the function of the variation of the curvature functions; $u = u(l)$ the function of the variation of the superelevation functions; and $v = v(l)$ where the function of variation of the velocity functions should be known to calculate the lateral change of acceleration [3].

1.2. Horizontal Geometry of Route Alignment

The horizontal geometry of alignment is created using plane curves with specific characteristics. These curves, classified with respect to their curvature functions, are:

a) First class curves: these curves have constant curvature such as line segment, and circular arc.
b) Second class curves: these curves have variable curvature such as clothoid, sinusoid and Bloss curve.

In addition, there are single spline curves that connect two straight lines without a circular arc [3].

There are some conditions on the connection points of two curves. These conditions are (see Fig. 3):

i. *Common connection point*: the coordinates of two curves must be equal on the connection points.
ii. *Common tangent*: the tangents of two curves must be equal on the connection points.
iii. *Equal radius of curvature*: the radii of the two curves must be equal on the connection points. Thus, discontinuities in the form of jump are eliminated in the curvature diagram.
iv. *Common tangent of curvature functions*: the first derivative of curvature functions of the two curves must be equal on the connection points. Thus, curvature functions have a common tangent on the connection points and thus eliminate the discontinuities in the form of a break.
v. *Equal radius of curvature on curvature functions*: the second derivative of the curvature functions of the two curves must be equal on the connection points. This condition is satisfied by only a few transition curves and is needed for extremely high speed railways.

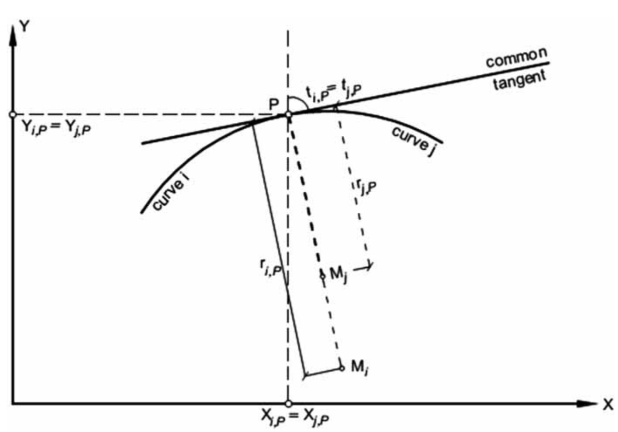

Fig. 3: The geometry of the connection point of two curves [3].

The first and second conditions are satisfied by all curves so they are visual conditions. However, the remaining conditions are special conditions required for higher speeds [3].

1.3. Organization

The rest of the paper is organized as follows. In Section 2, we briefly review the classical transition curve. Section 3 provides an overview of log-aesthetic curves. In Section 4, the curvature, superelevation and lateral change of the acceleration functions of the log-aesthetic curves are derived. We analyse the log-aesthetic curves with respect to their graphics of lateral change of acceleration in Section 5. In Section 6, we conclude our paper and present future recommendations.

2. CLASSICAL TRANSITION CURVE

In this section, the properties of the classical transition curve using the "clothoid" are given briefly.

2.1. Clothoid (Spiral Curve)

The clothoid has been studied extensively for many years. The main property of a clothoid is the linear change of curvature.

The coefficients of curvature function for a clothoid can be derived using $k(l) = a + bl$ with respect to the boundary conditions:

$$
\begin{aligned}
&k = 0 \;\; \mathit{for}\; l = 0 \;\mathit{and}\; l = L_1 + L_2 + L_3 \\
&k = k_{max} = 1/R \quad \mathit{for} \quad l = L_1 \;\mathit{and}\; l = L_1 + L_2 \\
&k' = 0 \quad \mathit{for\ all\ connection\ points}
\end{aligned}
\tag{4}
$$

The curvature function is obtained as $k(l) = l/LR$ using preceding boundary conditions where l is the horizontal length between O_i and O_i'.

Using the curvature function with respect to the geometry of combined curves (Fig. 4), we obtain:

$$k(l) = \begin{cases} \dfrac{l}{L_1 R}, & 0 \le l \le L_1 \\ \dfrac{1}{R}, & L_1 \le l \le L_1 + L_2 \\ \dfrac{(L_3 - l)}{L_3 R}, & L_1 + L_2 \le l \le L_1 + L_2 + L_3 \end{cases} \quad (5)$$

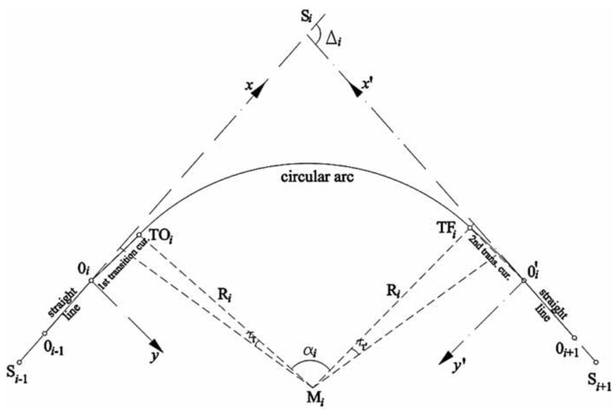

Fig. 4: The geometry of combined curves [3].

Superelevation functions are expected to have a similar structure to curvature functions [4].

$$k(l) = \begin{cases} \dfrac{u_{max} l}{L_1}, & 0 \le l \le L_1 \\ u_{max}, & L_1 \le l \le L_1 + L_2 \\ \dfrac{u_{max}(L_3 - l)}{L_3}, & L_1 + L_2 \le l \le L_1 + L_2 + L_3 \end{cases} \quad (6)$$

3. LOG-AESTHET*IC* CURVES

"Curves with logarithmic curvature graphs (LCGs) which are straight lines were called log-aesthetic curves, and they called curves with nearly straight LCGs quasi-log-aesthetic curves" [27]. These curve types can be used for aesthetic shape modelling and are expected to be an important part of computer aided design (CAD) systems. Log-aesthetic curves can also be used in the context of computer-aided aesthetic design (CAAD), in which the plot of curvature or the radius of curvature are the most significant factors for the quality of curves [35]. Log-aesthetic curves are defined as having a radius of curvature which is a function of their arc lengths [13]:

$$log\left(r\frac{dl}{dr}\right) = \alpha\ log(r) + c, \quad (7)$$

where the constant $c = -log\lambda$ and $(0, c)$ are the coordinates of the intersection of the y-axis with a line of a slope α, which is the shape parameter that determines the type of log-aesthetic curve (see Fig. 5).

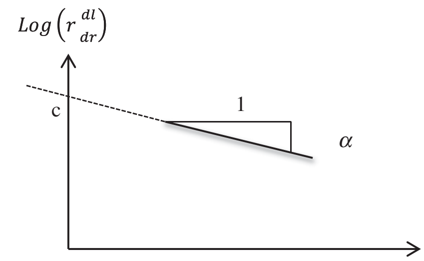

Fig. 5: Geometric meaning of shape parameter (α) in a logarithmic curvature graph.

After simply manipulating Eq. (7) and recollecting that c is a constant, we get:

$$\frac{dl}{dr} = \frac{r^{\alpha-1}}{\lambda} \quad (8)$$

Subsequently, after integrating Eq. (8) with respect to the radius of curvature r, the required equation of the log-aesthetic curve is found:

$$r(l) = \begin{cases} e^{\lambda l}, & \alpha = 0 \\ (\lambda a l + 1)^{1/\alpha}, & otherwise \end{cases} \quad (9)$$

where $\lambda = e^{-c}$, $0 < \lambda < \infty$. The following relation arising in the geometric interpretation of the curvature of any regular curve is well-known in differential geometry:

$$\kappa = \frac{1}{r} = \frac{d\theta}{dl} \quad (10)$$

In Geomatics engineering, one needs to obtain the tangent angle and X, Y coordinates of any curve to apply the points to the ground. Therefore, if Eq. (9) is substituted into (10), by integrating with respect to arc length, and setting $\theta = 0$ when $l = 0$, we obtain Eq. (11), and that relates the tangent angle θ with the arc lengths:

$$\theta(l) = \begin{cases} \dfrac{1 - e^{-\lambda l}}{\lambda} & \alpha = 0, \\ \dfrac{\log(\lambda l + 1)}{\lambda}, & \alpha = 1, \\ \dfrac{(\lambda \alpha l + 1)^{1-\frac{1}{\alpha}} - 1}{\lambda(\alpha - 1)}, & otherwise \end{cases} \quad (11)$$

Using Eqs. (8) and (10) we have:

$$\frac{d\theta}{dr} = \frac{dl}{rdr} = \frac{r^{a-2}}{\lambda} \quad (12)$$

Then the formulation of a log-aesthetic curve that relates the radius of curvature to the tangent angle is obtained by integrating Eq. (12) with respect to the tangent angle:

$$r(\theta)=\begin{cases} e^{\lambda\theta}, & \alpha=1 \\ ((\alpha-1)\lambda\theta+1)^{\frac{1}{\alpha-1}}, & otherwise \end{cases} \tag{13}$$

Finally, the parametric equations of a curve can be written as follows [31]:

$$x(\psi)=\int_0^{\psi} r(\theta)\cos\theta\, d\theta \tag{14}$$

$$y(\psi)=\int_0^{\psi} r(\theta)\sin\theta\, d\theta \tag{15}$$

After integrating Eqs. (14) and (15), the incomplete gamma function arises in parametric equations [31].

4. GEOMETRIC AND KINEMATIC FUNCTIONS OF LACs

In this section, the curvature and superelevation functions of LACs are calculated. Then, LCA functions of LACs are created.

4.1. Obtaining Curvature Functions

General curvature functions of LACs are obtained by using Eqs. (9) and (10):

$$k=(\lambda al+1)^{-1/\alpha}, \tag{16}$$

where λ parameter must be obtained in terms of R (see Chapter 2.1) in order to use Eq. (10) as a transition curve in route alignment. For $\alpha=1$, 2 and 3 values, the λ parameter is calculated separately with respect to the boundary conditions:

- For $\alpha=1$, the curvature function is $k=(\lambda l+l)^{-1}$

Applying $\kappa(0)$ and $\kappa(L)$, $\lambda=\frac{R-1}{L}$ is obtained and substituted (see Tab. 1 for details).

Shape parameter	Curvature functions
$\alpha=1$	$\kappa=\left(\frac{(R-1)l}{L}+1\right)^{-1}$ (17)
$\alpha=2$	$\kappa=\left(\frac{(R^2-1)l}{L}+1\right)^{-1/2}$ (18)
$\alpha=3$	$\kappa=\left(\frac{(R^3-1)l}{L}+1\right)^{-1/3}$ (19)

Tab. 1: Obtained curvature functions of LACs.

where 0 (zero) is minimum and L is the maximum arc length of the transition curve.

Eqs. (17), (18) and (19) do not satisfy $\kappa(0)=0$ boundary condition because $\kappa=(\lambda\alpha 0+1)^{-1/2}$ always results in 1 (one) independently from α and λ. Therefore, the general curvature function of LACs is assumed as $\kappa=(\lambda al+1)^{\frac{1}{\alpha}}+\gamma$ to eliminate this problem [8]. Then λ parameter can be recalculated as follows:

- For $\alpha=1$, the curvature function is $\kappa=(\lambda l+1)^{-1}+\gamma$

Applying $\kappa(0)$ and κ(L), $\gamma=-1$ and $\lambda=-\frac{1}{LR+L}$ is obtained then substituted.

Shape parameter	Curvature functions
$\alpha=1$	$\kappa=\left(\frac{-l}{LR+L}+1\right)^{-1}-1$ (20)
$\alpha=2$	$\kappa=\left(\frac{(-2.R-1)l}{L(R+1)^2}+1\right)^{-1/2}-1$ (21)
$\alpha=3$	$\kappa=\left(\frac{(R^3-1)l}{L}+1\right)^{-1/3}-1$ (22)

Tab. 2: Re-obtained curvature functions of LACs.

Curvature functions in Eqs. (20), (21), (22) (Tab. 2) tested for L=90 m and R=405 m are illustrated in Fig. 6.

Fig. 6a illustrates that the graphs of the curvature functions of LACs ($\alpha=1$, 2, 3) are unexpectedly the same. Therefore, the curvature function of LAC of $\alpha=10$ is obtained by using the same method and illustrated in Fig. 6b. If we compare the two graphs, they are almost same.

The graphs of the curvature functions of LACs have discontinuities in the form of breaks at connection points, therefore, the fourth condition (see Fig. 3) is not satisfied by LACs.

4.2. Obtaining Superelevation Functions

Superelevation functions should be expected to have a similar structure to curvature functions as they are obtained using the same method:

- For $\alpha=1$, the superelevation function is $u=(\lambda l+1)^{-1}+\gamma$ [8].

Applying $u(0)$ and $u(L)$, $\gamma=-1$ and $\lambda=-\frac{l}{LR+L}$ obtained and substituted into equation. Superelevation functions for different values of a shape parameter α are shown in Tab. 3.

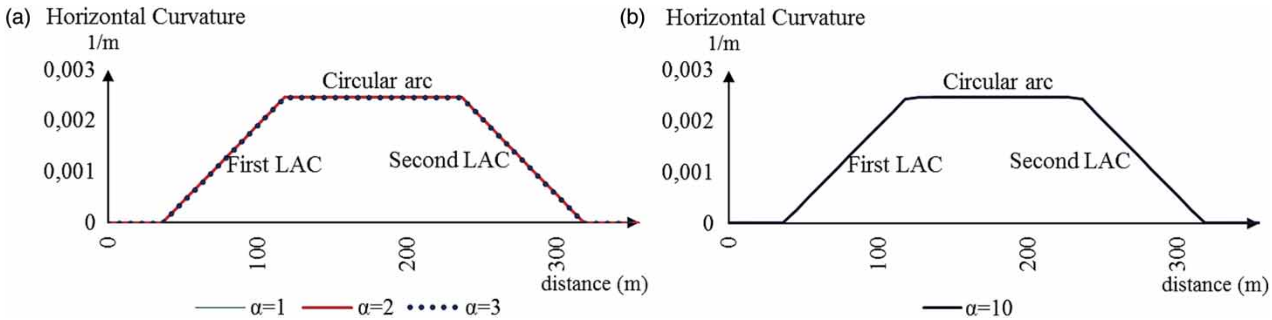

Fig. 6: (a) Graph of curvature functions of $\alpha = 1, 2, 3$. (b) Graph of curvature function of $\alpha = 10$.

Shape parameter	*Superelevation functions*	
$\alpha = 1$	$u(l) = \left(\frac{-lu}{Lu+L}+1\right)^{-1} - 1$	(23)
$\alpha = 2$	$u(l) = \left(\frac{-l(u^2+2u)}{L(u+1)^2}+1\right)^{-1/2} - 1$	(24)
$\alpha = 3$	$u(l) = \left(\frac{-l(u^3+3u^2+3u)}{L(u+1)^3}+1\right)^{-\frac{1}{3}} - 1$	(25)

Tab. 3: Obtained superelevation functions of LACs.

4.3. Obtaining Lateral Change of Acceleration Functions

The first derivatives of the curvature functions and superelevation functions must be derived to obtain the lateral change of the acceleration functions of LACs with respect to Eq. (3).

- For $\alpha = 1$, the functions are $u(l) = \left(\frac{-lu}{Lu+L}+1\right)^{-1} - 1$ and $\kappa(l) = \left(\frac{-l}{LR+L}+1\right)^{-1} - 1$,

Derived with respect to arc length (l): $\frac{\partial k(i)}{\partial l} = \frac{LR}{(-l+LR+L)^2}$ and $\frac{\partial u(i)}{\partial l} = \frac{u(Lu+L)}{(-lu+Lu+L)^2}$ obtained and substituted into Eq. (3). Lateral change of acceleration functions for different values of a shape parameter α are shown in Tab. 4.

5. IMPLEMENTATION AND ANALYSIS

The constant velocity motion model for railway project quantities is used to obtain the lateral change of acceleration values of LACs. In this model, velocity (v) is assumed to be constant $(a_T = 0)$ along the transition curve. To compare the LACs with a clothoid, the graphs of LCA functions of this model are illustrated in Fig. 7, taking into account the following magnitudes: minimum radius of curvature at points TO_i and TF_i is 405 m, the lengths of curves are $L = 300$ m, $L_1 = L_3 = 90\,m$ and $L_2 = 120\,m$, maximum superelevation

α	*Lateral change of acceleration functions*	
$\alpha = 1$	$z_{\alpha,1} = \frac{bv}{\sqrt{u^2+b^2}}\left(3ka_t + v^2\frac{LR}{(-l+LR+L)^2} - \frac{kv^2u+gb}{u^2+b^2}\frac{u(Lu+L)}{(-lu+Lu+L)^2}\right)$	(26)
$\alpha = 2$	$z_{\alpha,2} = \frac{bv}{\sqrt{u^2+b^2}}\left(3ka_t + v^2\frac{2.R+1}{2L(R+1)^2\left(\frac{(-2R-1)l}{L(R+1)^2}\right)^{3/2}} - \frac{kv^2u+gb}{u^2+b^2}\frac{u^2+2u}{2L(u+1)^2\left(\frac{-l(u^2+2u)}{L(u+1)^2}\right)^{3/2}}\right)$	(27)
$\alpha = 3$	$Z_{\alpha,3} = \left(3ka_3 + v^2\frac{3R^2+3R+1}{3L(R+1)^3}\left(\frac{(-3R^2-3R-1)}{L(R+1)^3}+1\right)^{4/2} - \frac{kv^2u+gb}{u^2+b^2}\frac{u^2+9u^2+3u}{3L(u+1)^2\left(\frac{-1(u^2+9u^2+3u)}{L(u+1)^2}+1\right)^{4/3}}\right)$	(28)

Tab. 4: Obtained lateral change of acceleration functions of LACs.

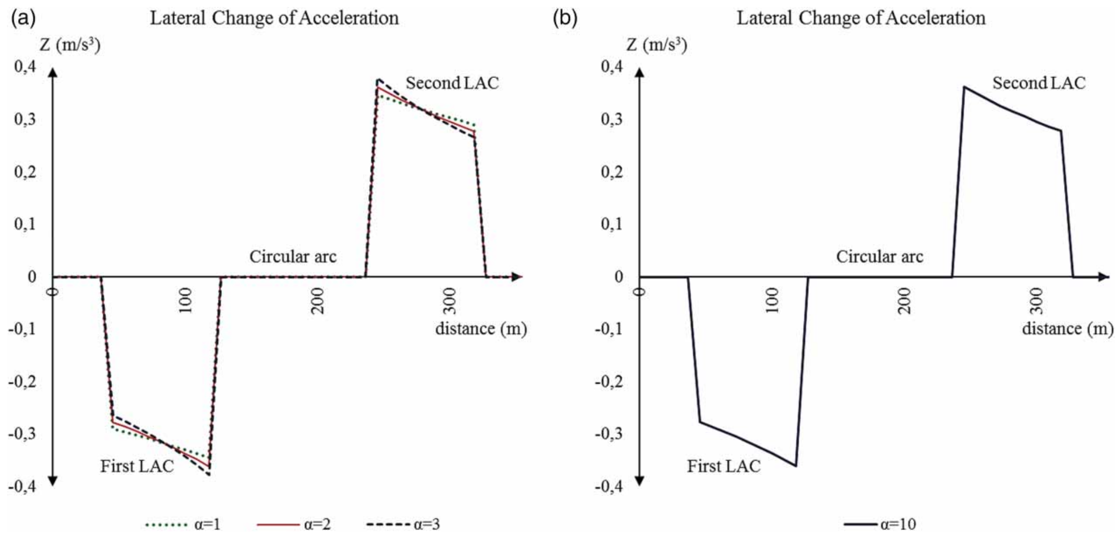

Fig. 7: (a) Graph of LCA functions of $\alpha = 1, 2, 3$. (b) Graph of LCA function of clothoid.

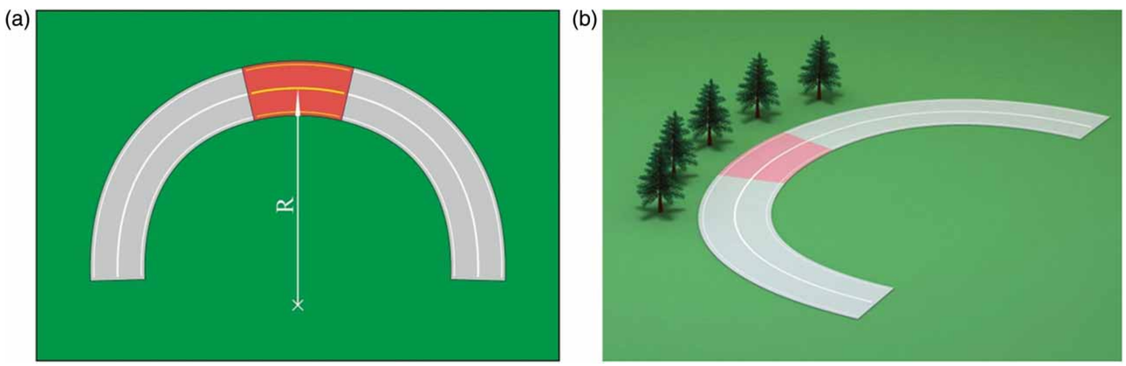

Fig. 8: Transition road modeling using our approach (here, the LACs' curve segments are in gray and the circular arc segment is in red) (for the interpretation of the references in order to color in this figure legend, the reader is referred to the web version of this article).

is $u_{MAX} = 0.15\,m$, horizontal width of the road platform (railway) is $b = 1.435$ m, and constant velocity is $v = 100\,km/h$ [26]. Some implementations of our approach are shown on Fig. 8 (a, b).

5.1. Analysing LACs with respect to LCA

The graphs of the lateral change of acceleration of the log-aesthetic curves are analysed with respect to:

- Continuity of the graphs of the lateral change of acceleration;
- Magnitudes of the graphs of the lateral change of acceleration.

The graphs of LCAs of LACs have discontinuities in the form of jumps, such as the diagrams of clothoid LCA. It is clear that the magnitude of any discontinuity at a point affects travel comfort, change of geometry, and wear. Thus, the results of the research into LACs illustrate similarities to the clothoid in terms of road vehicle kinematics.

The magnitudes of the diagrams of the lateral change of acceleration of log-aesthetic curves are about 0.26–0.38 m/sec^3 jumps at the connection points. The maximum magnitude of LCA values show differences of about 0.02 m/sec^3 depending on the shape parameter α value. The maximum values of the lateral change of acceleration are acceptable values with respect to the boundary values given in the literature.

5.2. Transition Curves in Road Design

Transition railroad modeling using our approach is presented on Fig. 9 (a, b).

6. SUMMARY AND FUTURE WORK RECOMMENDATION

In this study, different families of curves have been implemented as a transition curve in road design. The

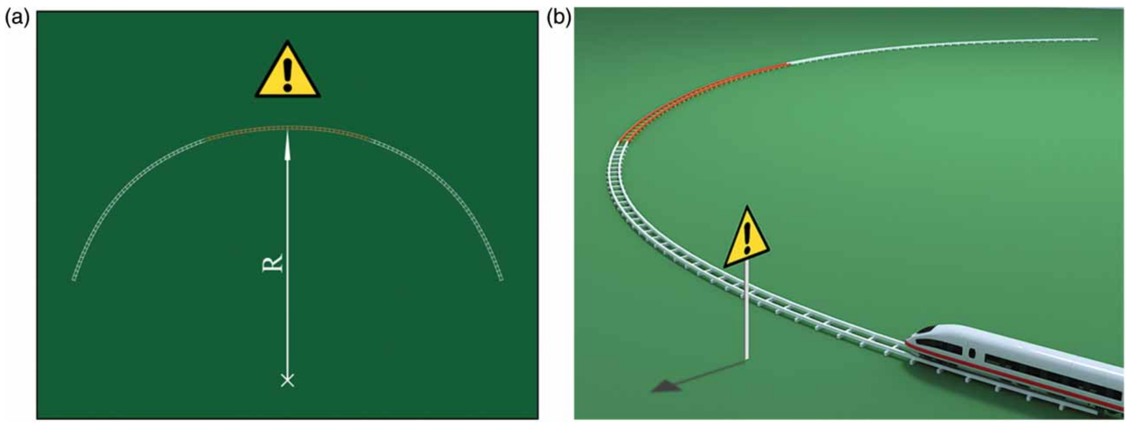

Fig. 9: Transition railroad modeling using our approach (here, the LACs' curve segments are in gray and the circular arc segment is in red). This approach cannot be used for movement at high speed of 200 km/h - 500 km/h (for the interpretation of the references in order to color in this figure legend, the reader is referred to the web version of this article).

transition curve-circular arc-transition curve combined model was used for implementation.

As a result of the analysis:

- The first, second and third conditions for connection points are satisfied;
- The fourth condition for connection points is not satisfied;
- The continuity of the graphs of the lateral change of acceleration is not satisfied;
- The maximum magnitudes for the lateral change of acceleration are satisfied by the log-aesthetic curves.

As a result, log-aesthetic curves can be implemented as a transition curve, but they do not have more specifications than the clothoid in terms of road vehicle kinematics. If they had many more specifications than the clothoid, we would create a new type of transition curve with better kinematical characteristics.

Many curves of the monotone curvature function, such as *superspirals* [30], *multispirals* [34], *Pythogorean-hodograph curves* [6], as well as *class A Bézier* [5] and *unit quaternion* curves [12], [10], [11], in addition to other high-quality curves [31], [32], have not been studied in terms of kinematics. There are possible opportunities for them to improve trends in highway design, especially in designing high-speed tracks, where not only kinematical properties but also the laws of technical aesthetics can be taken into account when looking to achieve the visual aesthetics [36,37]. All these issues will be the scope of our future work.

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