www.cadanda.com Computer-Aided Design and Applications

Development of Optimized Preliminary Vehicle Structural Model Using Simple Structural Beams-Frames (SSB) and Sub-Structural Analysis

Steven Tebby[1], Ahmad Barari[2] and Ebrahim Esmailzadeh[3]

[1]University of Ontario Institute of Technology, Steven.Tebby@uoit.ca
[2]University of Ontario Institute of Technology, Ahmad.Barari@uoit.ca
[3]University of Ontario Institute of Technology, Ebrahim.Esmailzadeh@uoit.ca

ABSTRACT

Optimum design of vehicle's structure is an important task in its development. The structure of a vehicle has complex interactions with the other vehicle components and has significant impact on the performance of the vehicle. Structural design is usually completed by a complex iterative process. The design changes at late design stages effect many other parameters in the design of vehicle. Therefore, it is highly valuable for designers to employ simple but effective analyses at the early design stages. A method called Simple Structural Beams-Frames (SSB) that uses Finite Element beam elements to represent the vehicle structure has been already developed in the previous work. However, the beam representation of the vehicle structure is not a complete preliminary model since the model is missing plates as the key structural elements. The beam model does not feature any planar sheets which are required prior to a detailed design process. In order to overcome this problem a substructure analysis and optimization procedure is implemented following the design of the beam frame structure.

Keywords: automotive structure design, finite element analysis, beam-frame model.

1. INTRODUCTION

The design of an automotive structure is critical to the overall performance of a vehicle. The structure of the vehicle is important to ensure it can satisfactorily carry the applied loads that occur [4]. The structure of a vehicle interacts with all other vehicle sub-components and has a complex influence one their functionality. Due to the structural design complexity, the design process is traditionally conducted by trial and error and is subject to numerous changes even in the latest stages of the design process. However, some of the changes in the design of structure may cause significant re-design of the other vehicle components and this may become very costly. Typically, it is much more desirable to maximize design changes during the early design stages and particularly before the detailed design activities [2]. However, employing a very comprehensive and detailed process of analyses at the conceptual design stage, when there is a greater range of design choices still available, may become very time consuming and computationally expensive as a result of the design freedom available. Therefore, it is very valuable for designers to employ simple but effective analyses at the early design stages. One option for conducting this analysis is through the use of a simplified vehicle structural model. The simplified model can be used to represent the geometric properties of the structure and then be analyzed and optimized in a numerical finite element program.

One of the most important criteria in automotive structural design is structural stiffness. The chassis stiffness, both in bending and torsion, has significant impacts on the ride and comfort characteristics as well as the overall dynamic vehicle performance [11, 12,18]. For this reason the stiffness values are used as design parameters to be optimized. Increasing the structural stiffness is highly critical in enhancing the vehicle's performance. However, due to economic constraints increasing the vehicle stiffness by increasing the structural weight is not recommended. An optimized solution is desired that maximizes structural stiffness while keeping the structural weight as low as possible.

Being able to efficiently analyze the body structure during the conceptual design stages is important to determining the performance characteristics. A primary method used to analyze the structure is the

Taylor & Francis
Taylor & Francis Group

method of Simple Structural Surfaces (SSS) [4], [3]. This method utilizes planar sheets to model the body structure. The SSS method can be used to determine the load-paths present in a body structure, but is unable to analyze an indeterminate static loading condition. Alternatively, the method utilized in this work, uses beam-frame elements to represent the structure as an equivalent space frame. The approach of using beam elements has the advantage of being able to determine displacements due to these forces by using the Finite Element Method (FEM). The use of the beam-frame finite element model can be used for basic analysis of a vehicle structure and as an initial estimate of some important vehicle parameters such as bending and torsion stiffness as well as some vibration characteristics. Using analogy of names, this method is referred to in this paper as the Simple Structural Beams-Frames (SSB) method. This paper presents an approach to develop an optimized preliminary vehicle structural model. The SSB model of a structure is a space frame representation and is therefore missing some planar sheet elements that are necessary in a typical vehicle structure such as the floor and roof.

An SSS model has been previously analyzed using commercial finite element software [3]. A diagram of this geometry is shown below in Fig. 1. The deflection results that were produced using this model are shown in Fig. 2.

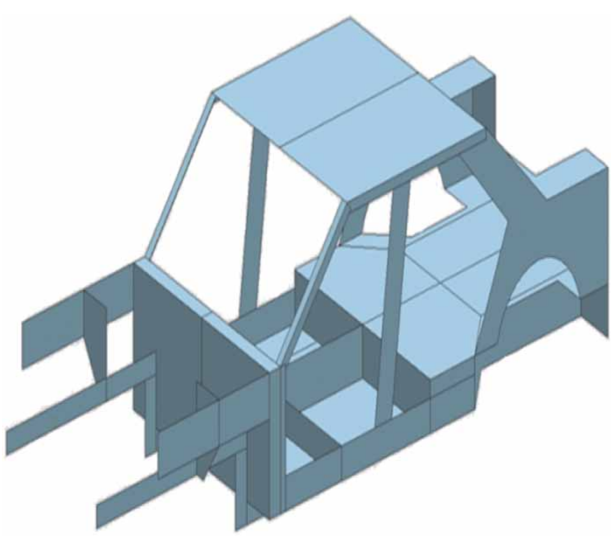

Fig. 1: SSS Model [3].

2. BACKGROUND

Both SSS and SSB methods can be used to determine forces that are present throughout the structure and assist with preliminary design decisions. A brief explanation of both of these analysis methods is presented here as background information. Another important aspect of this structural analysis is the utilized finite element method that is also briefly presented here.

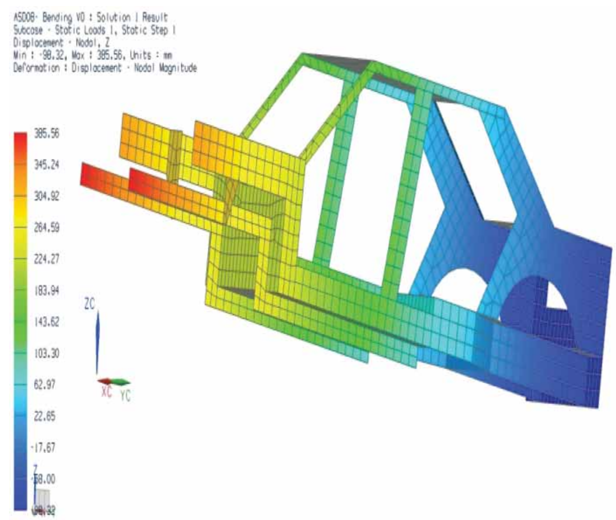

Fig. 2: Deflection of SSS Model [3].

The Simple Structural Surface method uses planar surfaces to model a structure. It was developed initially to analyze the load path of a vehicle [4]. The surfaces are able to react in plane loads only and transfer the forces from one surface to another via edge shear loads. The original intent of this method is to analyze the structure and determine a suitable load path. This method of analysis has a few limitations which restrict the benefits however as an initial estimation before the development of improved techniques it is sufficient. One of the major limitations of this method is that it cannot analyze structures with redundancies, which commonly occurs in an automotive structure, without utilizing the finite element method. This requires the structure to be statically determinant throughout, which may require removing a number of elements that may be required for a fully assembled vehicle. The second major limitation is this method does not have the capacity to determine deflections that will occur due to different loading conditions. This disadvantage prevents the method from significantly contributing to the design process since it doesn't allow an initial analytical estimation of some important design parameters such as stiffness. Overall the SSS method is only of interest as part of early automotive structural design and has been replaced by improved models that allow for a greater range of analysis such as the SSB method presented here. While the SSB method does require simplifications the structural model utilized is a better representation of modern space frame vehicles.

In order to overcome some of the limitations of the SSS method the SSB method is employed in this work. A beam-frame uses beam elements to model the structure of the vehicle [7,8,15]. An example of simple beam frame model is shown in Fig. 3. The beam frame model was developed primarily because it can be easily implemented in the FEM. This method allows the determination of the deflection of the vehicle based on applied loading conditions. Once the deflections have been found, it is possible to determine bending stiffness of the chassis. This method neglects the sheet components that occur in a structure however where necessary an extra beam element is implemented in the model to account for missing sheets [7]. The beam frame model also has the added flexibility of allowing for optimization of the design by improving the cross-section type and dimensions [8]. Finally the beam element model allows for the determination of the vibration characteristics [16].

More complete models have been developed that utilize plate and shell elements to more accurately model the vehicle structure [8]. However, their application may become too computationally expensive for an optimization process when there are many design variables. This is typically the case during the early stages of the design process. It is more appropriate to use a simplified conceptual model during the early optimization process to roughly select values for majority of the structural design parameters and then use the more accurate models for a few more important parameters and the final tuning during detail design. The SSB method presented here is a trade-off between accuracy and time, and is sufficient for the purposes of preliminary design estimation of majority of the design parameters. In order to improve the structural model a series of plates are added to the SSB structure following an optimization process.

The finite element solver developed for the SSB method uses typical beam elements with linear shape functions and Galerkin's Method is used for deriving the beam element equations [19]. The stiffness of an individual beam element being used in this work is shown below:

$$
K^e = \begin{bmatrix}
\frac{EA}{L} & 0 & 0 & 0 & 0 & 0 & \frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & \frac{6EI_z}{L^3} & 0 & \frac{-12EI_z}{L^3} & 0 & 0 & 0 & \frac{6EI_z}{L^3} \\
0 & 0 & \frac{12EI_z}{L^3} & 0 & \frac{-6EI_z}{L^2} & 0 & 0 & 0 & \frac{-12EI_z}{L^3} & 0 & \frac{-6EI_z}{L^2} & 0 \\
0 & 0 & 0 & \frac{GJ}{L} & 0 & 0 & 0 & 0 & 0 & \frac{-GJ}{L} & 0 & 0 \\
0 & 0 & \frac{-6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & 0 & 0 & 0 & \frac{6EI_z}{L^2} & 0 & \frac{2EI_y}{L} & 0 \\
0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L} & 0 & \frac{-6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_z}{L} \\
\frac{-EA}{L} & 0 & 0 & 0 & 0 & 0 & \frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-12EI_z}{L^3} & 0 & 0 & 0 & \frac{-6EI_y}{L^2} & 0 & \frac{12EI_y}{L^3} & 0 & 0 & 0 & \frac{-6EI_z}{L^2} \\
0 & 0 & \frac{-12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 & 0 & 0 & \frac{12EI_y}{L^3} & 0 & \frac{12EI_z}{L^3} & 0 \\
0 & 0 & 0 & \frac{-GJ}{L} & 0 & 0 & 0 & 0 & 0 & \frac{GJ}{L} & 0 & 0 \\
0 & 0 & \frac{-6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & 0 & 0 & 0 & \frac{12EI_z}{L^3} & 0 & \frac{4EI_y}{L} & 0 \\
0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_z}{L} & 0 & \frac{-6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L}
\end{bmatrix} \quad (2.1)
$$

In the above equation A is the cross-sectional area, I_z and I_y are the moments of inertia about the local z and y axes respectively, J is the polar moment of inertia, E is Young's modulus and is equal to 206×10^9 Pa G is the shear modulus and is equal to 79.8×10^9 Pa. The material properties being used are most closely related to carbon steel. As part of the substructure analysis the beam stiffness matrix needs to be combined with the plate stiffness's. The stiffness matrices for each component are combined according to the degrees of freedom they relate to. This is illustrated for a beam node below, where overlapping degrees of freedom have a summation of stiffness term and stiffness terms that are not part of the beam stiffness matrix are substituted into the sparse entries.

$$
K^1 = \begin{bmatrix}
\frac{EA}{L} + K_s(1,1) & K_s(1,2) & 0 & 0 & 0 & 0 \\
K_s(2,1) & \frac{12EI_z}{L^3} + K_s(2,2) & 0 & 0 & 0 & \frac{6EI_z}{L^2} \\
0 & 0 & \frac{12EI_y}{L^3} + K_b(1,1) & K_b(1,2) & \frac{-6EI_y}{L^2} + K_b(1,3) & 0 \\
0 & 0 & K_b(2,1) & \frac{GJ}{L} + K_b(2,2) & K_b(2,3) & 0 \\
0 & 0 & \frac{-6EI_y}{L^2} + K_b(3,1) & K_b(3,2) & \frac{4EI_y}{L} + K_b(3,3) & 0 \\
0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L}
\end{bmatrix} \quad (2.2)
$$

Fig. 3: Beam element model.

As can be seen the addition of the plate stiffness matrix to the existing beam stiffness matrix introduces a number of new terms with n representing the node being referred to, K_s referring to the plate shear stiffness matrix and K_b referring to the plate bending stiffness matrix. The plate shear and plate bending stiffness matrices are shown below. The numbers within the brackets refer to the position within the corresponding plate stiffness matrix. It should be noted that the summation of plate and beam stiffness matrices is also dependent on the connectivity's of the nodes and therefore a more general equation is not available. The stiffness matrices are calculated individually as, based on Kirchhoff's plate analysis, they are decoupled [24]. The stiffness matrices are formed in the same manner using the equation shown below.

$$[\mathbf{k}_{s,b}] = \int_{-1}^{1}\int_{-1}^{1} [\mathbf{B}_{s,b}]^{\mathrm{T}}[\mathbf{D}_{s,b}][\mathbf{B}_{s,b}]|\mathbf{J}|\,\mathrm{ds\,dt} \tag{2.3}$$

In the above equation the B matrix is based on the element shape functions and J is the Jacobian of the shape functions. The subscripts refer to the type of stiffness matrix with s representing shear and b representing bending. The element coordinate system is transferred to a isoparametric system in with s and t axes replacing x and y. the integration in the above equation can be very complex and is therefore generally done using two point Gaussian quadrature. The distinction between in plane and bending stiffness matrices is formed by the constitutive matrix, D. This matrix is formed based on the material properties and is shown below for each respective load condition.

$$[\mathbf{D}_s] = \frac{Et}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \tag{2.4}$$

$$[\mathbf{D}_b] = \frac{Et^3}{12(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \tag{2.5}$$

As can be seen the constitutive matrices are based on the modulus of elasticity (E), Poisson's ratio (ν) and the plate thickness (t). The in-plane constitutive matrix, shown in Eqn. (2.3), varies linearly with the plate thickness as would be expected since this would correspond to axial deformation. The bending constitutive matrix, shown in Eqn. (2.5), varies with the thickness cubed as is found in typical beam bending theory.

The stiffness matrix of a vehicle would be formed by assembling the individual element stiffness matrices into a global stiffness matrix using the direct stiffness method. The solution procedure is called the stiffness method where the displacements are unknown and related to the global forces by the stiffness matrix. The stiffness method is the most common solution method and is used in commercial finite element solvers. The FEM used here is a system of linear equations that can be solved using the developed computer program and implemented iteratively for the optimization process.

3. METHODOLOGY

The analysis and optimization of the beam-frame structure is a multi-step process. The first step is to determine appropriate loads to be applied to the structure. This is done by utilizing the existing analysis of a vehicle model based on the SSS method [3]. The SSB model is shown below, in Fig. 3, as it would appear in a commercial solver.

The optimization process is used to determine the beam section dimensions, which for the square beam element used here is the side length and thickness, which will give the smallest ratio between weight and stiffness. The optimization performed here is a multi-objective optimization algorithm that seeks to minimize the weight to bending stiffness and the weight to torsion stiffness of the structure. The optimization process uses a constrained multi-function process that utilizes four different algorithms. The algorithms used for the process are interior point, SQP, active set and trust region reflective [5,6,9,10,13,14,21–23]. The process is a goal attainment algorithm that attempts to minimize a non-linear multivariate function. The variables that can be adjusted are the unique beam section dimensions. The initial step in the optimization uses the set of uniform section properties that were found to give a sufficiently small displacement. The simultaneous optimization objective can be summarized as follows.

$$\text{Objective} = Min_{[a_1,t_1,a_2,t_2,\ldots,a_n,t_n]} \left\{ \frac{W}{K_T}, \frac{W}{K_B} \right\} \tag{3.1}$$

The output of the optimization process is the element section sizes that will give the smallest ratio between the structure weight and stiffness. The torsion and bending stiffness are combined individually with the weight to be objective functions to be minimized simultaneously. Bounds are set for the section values based on the initial analysis so that the values being chosen are physically realistic. The process uses a Hessian to drive the direction of each step and the process ends when a set number of consecutive trials show no improvement to within a specified tolerance.

Following the optimization of the beam structure the substructure analysis procedure begins. The substructure analysis is used to determine suitable plate thickness values. The same optimization algorithm is utilized to ensure that the addition of the plate components will not substantially increase the overall structure weight. Substructure analysis is implemented as a limitation of the codes being used and future work would combine the plate components into the SSB structural model to form an enhanced SSB model that can have the same optimization performed. The substructures consist of a plate component bordered by a series of beam elements as shown below in Fig. 4. The loading condition is based on internal loads of the optimized beam structure and is applied as though the plate/beam substructure is a cantilevered system.

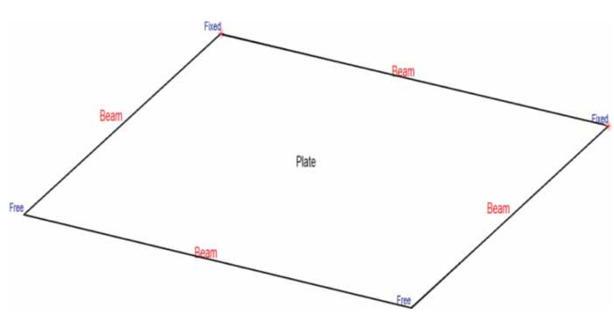

Fig. 4: Plate/Beam substructure.

The thicknesses of the plate components are determined through an optimization procedure that is implemented to ensure the weight increase of the structure is minimal. In order to have greater flexibility in the optimization the beam element section sizes are able to be reduced slightly. Any increases in deformation due to this reduction should be compensated for by the addition of the plate component. The substructures that need to be analyzed are illustrated below in Fig. 5.

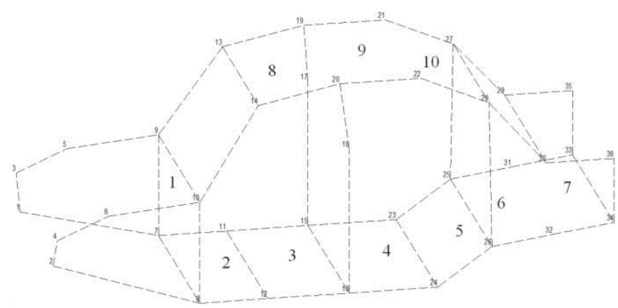

Fig. 5: Vehicle substructures.

4. IMPLEMENTATION

The presented method has been implemented for a vehicle case study. The optimization of the structure has been previously performed and only the results of the optimization process are presented here. The loading conditions for bending and torsion are shown below.

The first step of the process was to determine initial loads. These loads are based on assumed loads that are commonly found in a vehicle such as passengers, the power train and the other components. The structure loads are found based on existing analysis of the SSS method and can be found in the Tab. 1.

Component	Weight (N)	Centre of Gravity Position (m)
Front Bumper	200	0
Powertrain	3000	0.65
Front Passengers & Seats	2000	2.2
Rear Passengers & Seats	2500	3
Fuel Tank	500	2.95
Luggage	950	4
Rear Bumper	300	4.4
Exhaust	350	2.5
Front Structure	2227.5	0.675
Passenger Compartment	3870	2.425
Rear Structure	1170	3.95

Tab. 1: Component weights for initial analysis.

The final step in the procedure is to perform the optimization. As stated the optimization is a nonlinear constrained optimization that seeks to minimize the ratio between the weight and the stiffness.

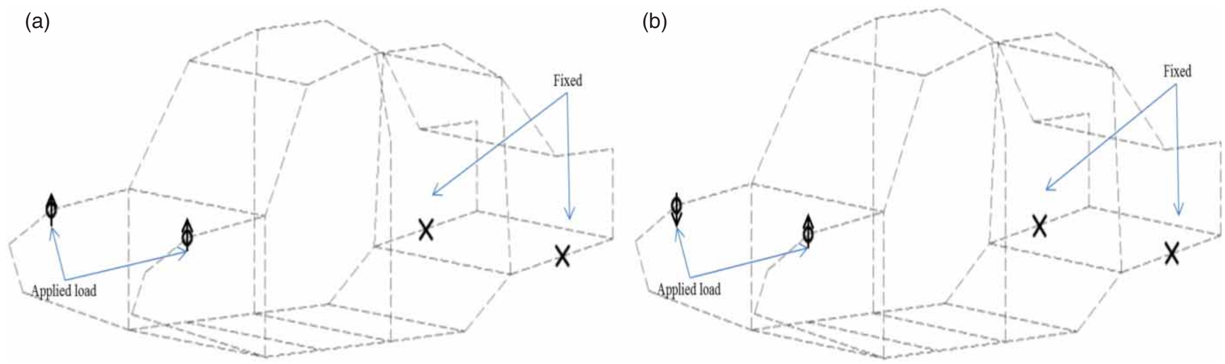

Fig. 6: Beam element geometry with constraints and loads: (a) bending, and (b) torsion.

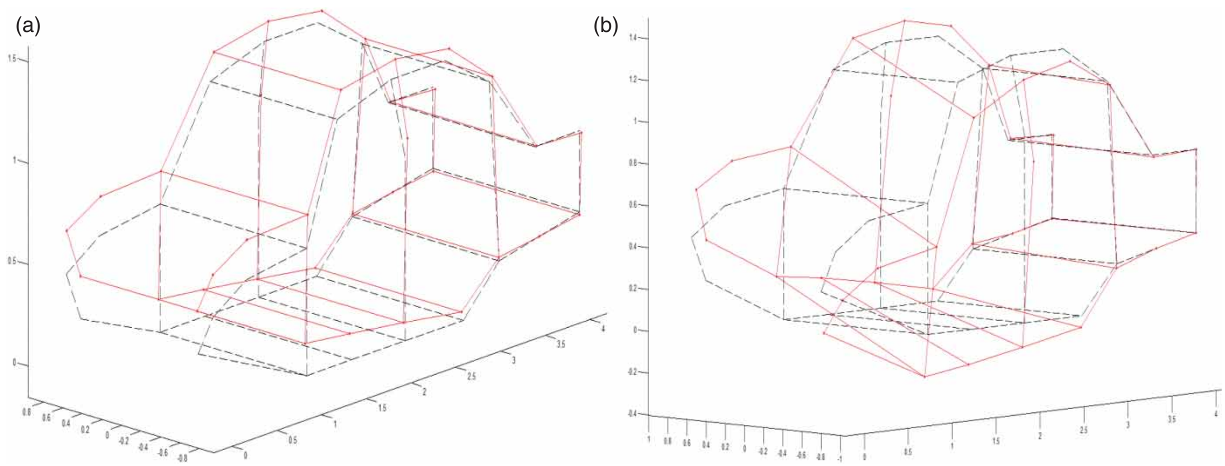

Fig. 7: Deflection of vehicle model: (a) bending loads, and (b) torsion loads.

The initial condition for the optimization process was the radius values that resulted in an acceptable static deflection of less than 15 mm. After running the optimization process the following results were found.

As can be seen the optimization process resulted in a substantial reduction in structure weight with slight increases in both stiffness values. This process results in a reduction in both objective functions. The optimization convergence and Pareto front can be seen below in Fig. 8, Fig. 9 and Fig. 10.

The images above show the convergence of both objective functions as the optimization proceeds and as can be seen both objective functions achieve convergence based on the tolerance specified. The Pareto Front shown in Fig. 10 illustrates the interaction of the objective functions throughout the optimization process. With the optimization process complete the next stage in the process can begin

The initial step of this process is calculating the internal loads of the optimized beam structure under static loading. The internal loads are used for the loading condition of a cantilevered substructure as illustrated in Fig. 3 and are applied at the free end. The optimization objectives are to minimize the substructure weight as well as the magnitude of the deflections at the free end. A summary of the plate thicknesses, as well as the final beam element dimensions are shown below in Tab. 2.

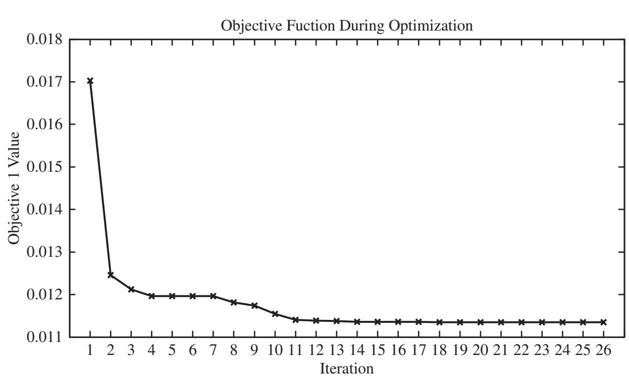

Fig. 8: Convergence of first objective function.

As can be seen the addition of the plates did reduce the size of some beam elements with the greatest changes occurring in the wall thickness. These results conclude the process of substructure analysis and represent a complete vehicle structural model.

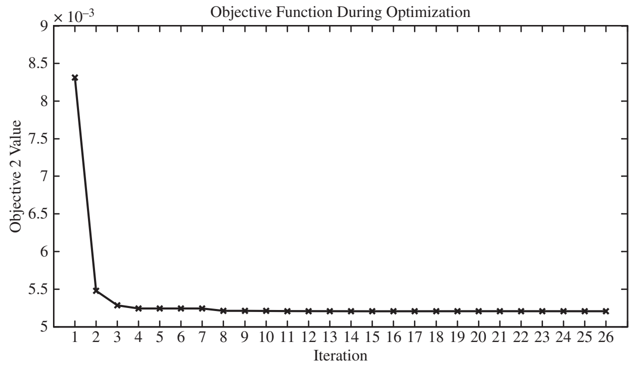

Fig. 9: Convergence of second objective function.

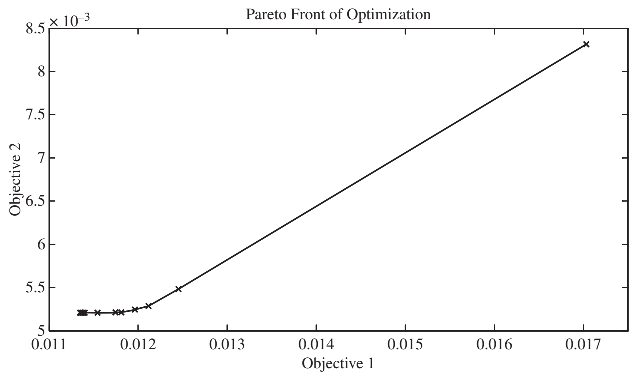

Fig. 10: Pareto front for optimization process.

5. CONCLUSION

A new approach to developing a preliminary vehicle structural model is presented in this work. The structural model is developed based on existing beam element structures. The beam element structure is subjected to an optimization process, such as occurs in the initial stages of the design process when the greatest design freedom is available. The optimization sought to minimize the ratio between the structure weight and structure stiffness, with specific attention paid to weight reduction. The beam structure utilizes hollow square sections with the side length and wall thickness chosen as design variables. Following

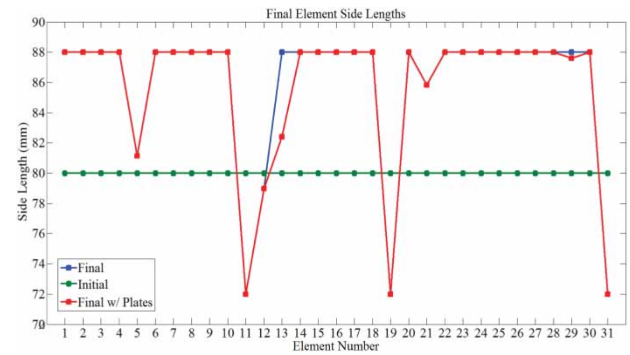

Fig. 11: Final element side lengths.

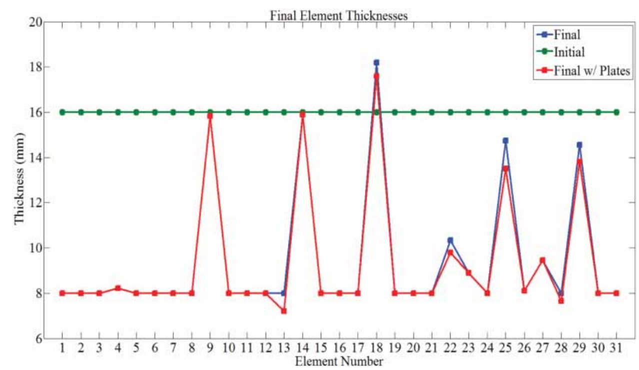

Fig. 12: Final element thicknesses.

Plate	Thickness (mm)
1	0.620752946
2	0.60706
3	0.608724024
4	0.621295302
5	0.60706
6	0.60706
7	0.60706
8	0.60706
9	0.60706
10	0.60706

Tab. 3: Plate thicknesses.

Parameter	Initial Values	Optimized Values	% Improvement
Weight, (N)	12567.91	8702.85	−30.7534
Bending Stiffness, (N/m)	738024.93	767037.35	3.9311
Torsion Stiffness, (Nm/rad)	1511743.86	1671002.90	10.5348
Objective 1	0.01702912	0.01134606	−33.3726
Objective 2	0.00831352	0.00520816	−37.3531
Attainment Factor		0.0052	

Tab. 2: Results of optimization process.

the optimization of the beam element model a process of substructure analysis is employed to estimate suitable plate thicknesses. To accommodate the addition of plate components the beam element section dimensions of each substructure were given an allowable reduction. Each substructure was subjected to the same optimization algorithm to ensure a minimal increase in overall structure weight.

ACKNOWLEDGEMENT

The research support provided by the Natural Science and Engineering Research Council Of Canada (NSERC) is greatly appreciated.

REFERENCES

[1] Antoniou, A.; Lu, Wu-Sheng.: Practical Optimization Algorithms and Engineering Applications, Springer Science, Victoria, Canada, 2007.

[2] Barari, A.; Platanitis, G.; Pop-Iliev, R.: Introducing Design Rigidity to Model Unexpected Disturbances In a DSM-Based Design Process, ASME 2010 IDETC-CIE, 2010.

[3] Barari, A.; Esmailzadeh, E.; Tebby, S.: Methods to Determine Torsion Stiffness in an Automotive Chassis, Computer Aided Design and Applications, PACE (1), 2011, 66-75

[4] Brown, Jason C.; Robertson, John A.; Serpento, Stan T.: Motor Vehicle Structures, Butterworth Heinemann 2002.

[5] Byrd, R. H.; Gilbert, J. C.; Nocedal, J.: A Trust Region Method Based on Interior Point Techniques for Nonlinear Programming, Mathematical Programming, 89, 2000, 149–185. http://dx.doi.org/10.1007/PL00011391

[6] Byrd, R. H., Hribar, Mary E.; Nocedal, Jorge: An Interior Point Algorithm for Large-Scale Nonlinear Programming, SIAM Journal on Optimization, SIAM Journal on Optimization, 9, 1999, 877–900. http://dx.doi.org/10.1137/S1052623497325107

[7] Ceglarek, D.; Shi, J.; Shiu, B. W.: Flexible Beam-Based Modeling of Sheet Metal Assembly for Dimensional Control, Transactions of NAMRI, 25, 1997.

[8] Chon, Choon T.; El-Essawi, Mohamed; Mohammadtorab, Homa: Generic Stick Model of a Vehicular Structure, Proceedings Society of Automotive Engineers, 1986, 235-241. http://dx.doi.org/10.4271/860825

[9] Coleman, T. F.; Li, Y.: An Interior, Trust Region Approach for Nonlinear Minimization Subject to Bounds, SIAM Journal on Optimization, 6, 1996, 418–445. http://dx.doi.org/10.1137/0806023

[10] Coleman, T. F.; Li, Y.: On the Convergence of Reflective Newton Methods for Large-Scale Nonlinear Minimization Subject to Bounds, Mathematical Programming, 67, 1994, 189–224. http://dx.doi.org/10.1007/BF01582221

[11] Crocombe, A.; Sampe, E.; Somiotti, A.: Chassis Torsional Stiffness: Analysis of the Influence on Vehicle Dynamics, SAE 2010 World Congress & Exhibition, 2010.

[12] George, A. R.; Riley W. B.: Design, Analysis and Testing of a Formula SAE Car Chassis, SAE Motorsports Engineering Conference and Exhibition, 2002.

[13] Gill, P. E., Murray W.; Wright, M. H.: Practical Optimization, London, Academic Press, 1981

[14] Han, S. P.: A Globally Convergent Method for Nonlinear Programming, Journal of Optimization Theory and Applications, 22, 1977, 297. http://dx.doi.org/10.1007/BF00932858

[15] Happian-Smith, Julian: An Introduction to Modern Vehicle Design, Butterworth-Heinemann, United States of America 2002

[16] Kamal, Mounir M.; Wolf Jr., Joseph A.: Finite Element Models for Autmotoive Vehicle Vibrations, Finite Element Applications in Vibration Problems, 1977, 67-92

[17] Lange, Kenneth: Optimization, Springer-Verlag, Los Angeles, United States of America, 2004

[18] Law, E. H.; Raju, S.; Soni, P. H.; Thompson, L. L.: The Effects of Chassis Flexibility on Roll Stiffness of a Winston Cup Car, Motorsports Engineering Conference Proceedings, 1998

[19] Logan, Daryl L.: A First Course in the Finite Element Method, Thomsen Canada Limited, Toronto, Canada 2007

[20] Malen, Donald E.: Fundamentals of Automobile Body Structure Design, SAE, Warrendale, United States of America 2011 http://dx.doi.org/10.4271/R-394

[21] Morales, J. L.; Nocedal, J.; Orban, D. Waltz, R. A.: An interior algorithm for nonlinear optimization that combines line search and trust region steps, Mathematical Programming, Vol 107, 2006, pp. 391–408 http://dx.doi.org/10.1007/s10107-004-0560-5

[22] Powell, M. J. D.: The Convergence of Variable Metric Methods For Nonlinearly Constrained Optimization Calculations, Nonlinear Programming 3, Academic Press, 1978

[23] Powell, M. J. D.: A Fast Algorithm for Nonlinearly Constrained Optimization Calculations, Numerical Analysis, ed. G.A. Watson, Lectures in Mathematics, Springer Verlag, 630, 1978

[24] Reddy, J. N.: Theory and analysis of elastic plates and shells, 2nd ed. CRC Press, Boca Raton, United States of America, 2007.