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Shape Analysis of Cubic Bézier Curves – Correspondence to Four Primitive Cubics

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ABSTRACT

In this paper, we show that any planar polynomial cubic Bézier curve can be described as an affine transformation of a part of four primitive cubics ($x^3 + x^2 - 3y^2 = 0, x^3 - 3y^2 = 0, x^3 - x^2 - 3y^2 = 0$, and $x^3 - y = 0$), and propose an algorithm to derive the transformation matrix.
For a given cubic Bézier curve, we derive the linear moving line that follows the curve, and find the double point $\boldsymbol{D}$ of the curve. If $\boldsymbol{D}$ is not a point at infinity, we derive the quadratic parallel moving line that follows the curve. By testing whether the quadratic moving line crosses $\boldsymbol{D}$ or not, we classify the curve into three cases (*crunode, cusp*, or *acnode*) which correspond to three primitive cubics. If $\boldsymbol{D}$ is a point at infinity, the curve is classified as fourth case (*explicit cubic*), which requires exceptional process. For each case, the affine transformation matrix between the primitive cubic and given Bézier curve can be derived. We confirm that the proposed algorithm never fails unless the cubic Bézier curve is degree reducible or consists of four collinear control points.

Keywords: moving line, double point, crunode, cusp, acnode, inflection point

1. INTRODUCTION

For highly aesthetic curve design, it is important to control curvature properties. In addition to tangent and curvature continuities, a fair curve should have the property that its curvature varies monotonically [5]. As a higher level property, Harada *et al.* analyzed many existing aesthetic curves in real objects, and found that the *logarithmic curvature histogram* (or *logarithmic curvature graph*, LCG) of such aesthetic curve tends to be linear [6]. Yoshida *et al.* proposed an interactive method to control a curve segment with linear LCG [12]. Such kind of curves, called *log-aesthetic curves*, has potential to be utilized in future CAD system, especially in initial design stage. For sophisticated CAD operations, however, log-aesthetic curves are not suitable as in original form, and need to be approximated with conventional parametric curves [13]. Therefore, higher level curvature analysis is required for parametric curves.

However, analysis of curvature variation of parametric curves is quite difficult. For example, as the simplest subject in such analysis, monotonic curvature condition of planar cubic Bézier curve has been studied. Walton *et al.* proposed a selected family of cubic Bézier curves that guaranties monotonic curvature [10]. Dietz presented G^1 Hermite interpolation condition for polynomial (*i.e.* non-rational) cubic Bézier curves by experimental analysis [3] and extended it to rational cubic Bézier [4]. Unfortunately, these results are not intuitive. This is due to the complicated relationship between geometry of control points and curvature variation of the curve.

In our project, we plan to investigate the curvature variation properties for the *whole set* of planar polynomial cubics instead of curve segments. Here, the whole set consists of limited number of primitive cubics with affine transformation. By investigation of each primitive cubic and its scaling and skewing, curvature properties of cubic curve segments can be surveyed completely. In this paper, as the preparation of the project, we clarify the correspondence between Bézier curves and the four primitive cubics: $x^3 + x^2 - 3y^2 = 0, x^3 - 3y^2 = 0, x^3 - x^2 - 3y^2 = 0$, and $x^3 - y = 0$. First, we classify cubic Bézier curves into four cases by checking the singularities. Then, we show that the whole set of cubic Bézier curves is equivalent to the four primitive cubics and their affine transformation, which is our main contribution.

2. RELATED WORKS

It is well-known that parametric curve segments may have inflection points, loops and cusps. These are

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usually undesirable features for curve design. Several studies have been done to derive conditions of these features and/or classification of curves, especially for cubic curve segments [7,9,11]. For example, Stone *et al.* presented the geometric condition of Bézier control points whether inflection points or singularities exist in the curve segment [9]. On the other hand, we focus on whole cubic curves instead of curve segments, find singularities, classify them into four categories, and show the correspondence to four primitive cubics.

Our research is also related to ideal basis of planar curves. Sederberg *et al.* presented that a cubic Bézier curve is intersection of two moving lines that are linear and quadratic, and a quartic Bézier curve is intersection of two quadratic moving lines (or linear one and cubic one in special case) [8]. Cox *et al.* generalized this idea as the ideal basis of rational curves, which is named μ-*basis* [2]. They showed that the ideal of any degree n planar rational curve can be generated by two polynomials that are degree n_1 and n_2 in t, where $n_1 + n_2 = n$. Chen *et al.* proposed an efficient algorithm to compute a μ-basis of a rational curve [1]. Our proposed method is an application of μ-basis, and we focus on polynomial cubic curves. For each curve, we find a specific μ-*basis* that enables to clarify the correspondence to the primitive cubics.

3. MOVING LINES

In this paper, most part of the theory is based on moving line (family of lines on Bernstein basis) [8]. Here are some definitions and properties of point, line, Bézier curve, and moving line in 2D homogeneous coordinate.

- Point: $\boldsymbol{P} = (X\ Y\ W) = w\ (x\ y\ 1)$.
- Line: $\boldsymbol{L} = (a\ b\ c):\ aX + bY + cW = 0$.
- Point $\boldsymbol{P}$ lies on line $\boldsymbol{L} : \boldsymbol{P} \cdot \boldsymbol{L} = 0$.
- Line $\boldsymbol{L}$ contains two points $\boldsymbol{P}_0$ and $\boldsymbol{P}_1 : k\boldsymbol{L} = \boldsymbol{P}_0 \times \boldsymbol{P}_1$, where k is a constant.
- Point $\boldsymbol{P}$ is the intersection of two lines $\boldsymbol{L}_0$ and $\boldsymbol{L}_1 : k\boldsymbol{P} = \boldsymbol{L}_0 \times \boldsymbol{L}_1$.
- Bézier curve $\boldsymbol{P}(t)$ (degree n): $\boldsymbol{P}(t) = \sum_{i=0}^{n} B_i^n(t)\boldsymbol{P}_i$, where $\boldsymbol{P}_i (0 \le i \le n)$ are control points, and $B_i^n(t) = \binom{n}{i} (1-t)^{n-i} t^i$.
- Moving line $\boldsymbol{L}(t)$ (degree n): $\boldsymbol{L}(t) = \sum_{i=0}^{n} B_i^n(t)\,\boldsymbol{L}_i$, where $\boldsymbol{L}_i (0 \le i \le n)$ are control lines.
- Curve $\boldsymbol{P}(t)$ follows moving Line $\boldsymbol{L}(t)$ (*or* moving line $\boldsymbol{L}(t)$ follows curve $\boldsymbol{P}(t)$), *i.e.* for all t, the point $\boldsymbol{P}(t)$ lies on the line $\boldsymbol{L}(t) : \boldsymbol{P}(t) \cdot \boldsymbol{L}(t) = 0$.
- Moving line $\boldsymbol{L}(t)$ follows two curves $\boldsymbol{P}_0(t)$ and $\boldsymbol{P}_1(t) : k(t)\boldsymbol{L}(t) = \boldsymbol{P}_0(t) \times \boldsymbol{P}_1(t)$, where $k(t)$ is a scalar function of t.
- Curve $\boldsymbol{P}(t)$ is the intersection of two moving lines $\boldsymbol{L}_0(t)$ and $\boldsymbol{L}_1(t) : k(t)\,\boldsymbol{P}(t) = \boldsymbol{L}_0(t) \times \boldsymbol{L}_1(t)$.

4. CLASSIFICATION OF CUBIC BÉZIER CURVES

In this section, for given four control points $\boldsymbol{P}_0, \boldsymbol{P}_1, \boldsymbol{P}_2$ and $\boldsymbol{P}_3$ on a plane, we show that the Bézier curve $\boldsymbol{P}(t)$:

$$\boldsymbol{P}(t) = (1-t)^3\boldsymbol{P}_0 + 3(1-t)^2 t\boldsymbol{P}_1 + 3(1-t)t^2\boldsymbol{P}_2 + t^3\boldsymbol{P}_3 \tag{4.1}$$

can be described as an affine transformation of a part of four primitive cubics ($x^3 + x^2 - 3y^2 = 0, x^3 - 3y^2 = 0, x^3 - x^2 - 3y^2 = 0$, and $x^3 - y = 0$) shown in Fig. 1. Here, any cubic Bézier curve is acceptable unless it is degenerate; *i.e.* if the curve satisfies the following conditions:

- It is not degree reducible;
- All of its four control points are not collinear.

Selection of these four primitive cubics will be discussed in subsection 5.3.

The outline of the algorithm is as follows. First, we derive the linear moving line $\boldsymbol{L}_0(t)$ that follows the Bézier curve $\boldsymbol{P}(t)$, and find the double point $\boldsymbol{D}$ of the curve. If $\boldsymbol{D}$ is not a point at infinity, we derive the quadratic parallel moving line $\boldsymbol{L}_1(t)$ that follows the curve $\boldsymbol{P}(t)$. By testing whether the moving line $\boldsymbol{L}_1(t)$crosses the double point $\boldsymbol{D}$ or not, we classify the curve into three cases (*crunode*, *cusp*, or *acnode*). Each case corresponds to each primitive cubic ($x^3 + x^2 - 3y^2 = 0, x^3 - 3y^2 = 0$, or $x^3 - x^2 - 3y^2 = 0$). If $\boldsymbol{D}$ is a point at infinity, the curve is classified as fourth case (*explicit cubic*) and corresponds to the primitive cubic $x^3 - y = 0$. In this case, we derive the quadratic moving line $\boldsymbol{L}_2(t)$ that contains the inflection point $\boldsymbol{I}$ of the curve. For each case, by using the correspondence of three reference points ($\boldsymbol{D}$ or $\boldsymbol{I}$, and two other characteristic points), from which we obtain the affine transformation matrix $\mathbf{M}$ between the primitive cubic and given Bézier curve $\boldsymbol{P}(t)$.

4.1. Double Point and Linear Moving Line

A moving line $\boldsymbol{L}(t)$ that goes through a fixed point $\boldsymbol{K}$ and follows the curve $\boldsymbol{P}(t)$ can be obtained as,

$$\boldsymbol{L}(t) = \boldsymbol{K} \times \boldsymbol{P}(t). \tag{4.2}$$

This is generally a cubic moving line. However, if the fixed point is on the curve, *i.e.* $\boldsymbol{K} = \boldsymbol{P}(\tau)$, the moving line $\boldsymbol{L}(t)$ has a common factor and is substantially quadratic [8]:

$$\begin{aligned}\boldsymbol{L}(t) = \boldsymbol{P}(\tau) \times \boldsymbol{P}(t) &= (t-\tau)\left[(1-\tau)^2 \quad (1-\tau)\tau \quad \tau^2\right] \\ &\times \begin{bmatrix} \boldsymbol{Q}_0 \times \boldsymbol{Q}_1 & \boldsymbol{Q}_0 \times \boldsymbol{Q}_2 & \boldsymbol{Q}_0 \times \boldsymbol{Q}_3 \\ \boldsymbol{Q}_0 \times \boldsymbol{Q}_2 & \boldsymbol{Q}_0 \times \boldsymbol{Q}_3 + \boldsymbol{Q}_1 \times \boldsymbol{Q}_2 & \boldsymbol{Q}_1 \times \boldsymbol{Q}_3 \\ \boldsymbol{Q}_0 \times \boldsymbol{Q}_3 & \boldsymbol{Q}_1 \times \boldsymbol{Q}_3 & \boldsymbol{Q}_2 \times \boldsymbol{Q}_3 \end{bmatrix} \\ &\times \begin{bmatrix} (1-t)^2 \\ (1-t)t \\ t^2 \end{bmatrix}\end{aligned} \tag{4.3}$$

where $\boldsymbol{Q}_0 = \boldsymbol{P}_0, \boldsymbol{Q}_1 = 3\boldsymbol{P}_1, \boldsymbol{Q}_2 = 3\boldsymbol{P}_2, \boldsymbol{Q}_3 = \boldsymbol{P}_3$.

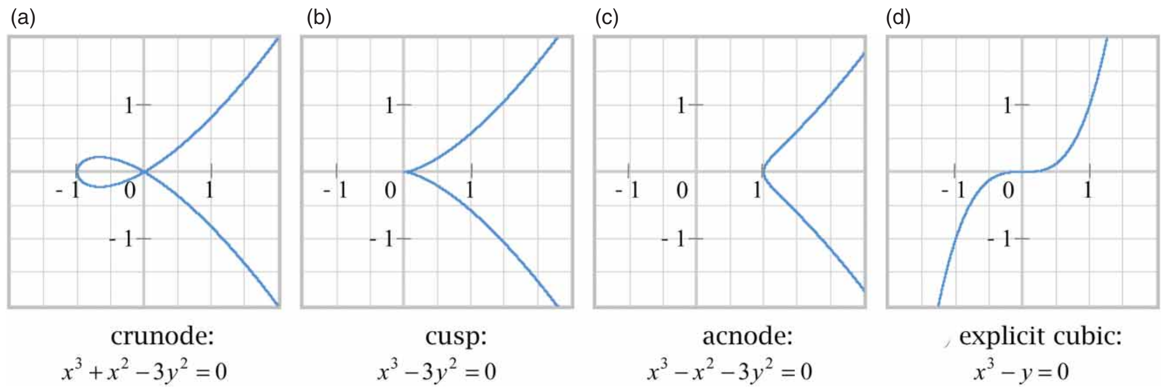

Fig. 1: Four Primitive cubics.

Furthermore, if $\boldsymbol{P}(\tau)$ is the double point of the cubic curve, $\boldsymbol{L}(t)$ becomes substantially linear [8].

In order to classify the Bézier curve $\boldsymbol{P}(t)$, we first find such linear moving line that follows the curve. In Eqn. (4.3), $\boldsymbol{L}(t)$ follows $\boldsymbol{P}(t)$. Thus,

$$\begin{aligned}&[(1-\tau)^2 \quad (1-\tau)\tau \quad \tau^2] \\ &\times \begin{bmatrix} \boldsymbol{Q}_0\times\boldsymbol{Q}_1 & \boldsymbol{Q}_0\times\boldsymbol{Q}_2 & \boldsymbol{Q}_0\times\boldsymbol{Q}_3 \\ \boldsymbol{Q}_0\times\boldsymbol{Q}_2 & \boldsymbol{Q}_0\times\boldsymbol{Q}_3+\boldsymbol{Q}_1\times\boldsymbol{Q}_2 & \boldsymbol{Q}_1\times\boldsymbol{Q}_3 \\ \boldsymbol{Q}_0\times\boldsymbol{Q}_3 & \boldsymbol{Q}_1\times\boldsymbol{Q}_3 & \boldsymbol{Q}_2\times\boldsymbol{Q}_3 \end{bmatrix} \\ &\times \begin{bmatrix}(1-t)^2\\(1-t)t\\t^2\end{bmatrix}\cdot\boldsymbol{P}(t)=0. \end{aligned} \tag{4.4}$$

Since Eqn. (4.4) is satisfied for any τ,

$$\begin{cases} \Big((\boldsymbol{Q}_0\times\boldsymbol{Q}_1)(1-t)^2+(\boldsymbol{Q}_0\times\boldsymbol{Q}_2)(1-t)t \\ \quad +(\boldsymbol{Q}_0\times\boldsymbol{Q}_3)t^2\Big)\cdot\boldsymbol{P}(t)=0 \\ \Big((\boldsymbol{Q}_0\times\boldsymbol{Q}_2)(1-t)^2+(\boldsymbol{Q}_0\times\boldsymbol{Q}_3+\boldsymbol{Q}_1\times\boldsymbol{Q}_2)(1-t)t \\ \quad +(\boldsymbol{Q}_1\times\boldsymbol{Q}_3)t^2\Big)\cdot\boldsymbol{P}(t)=0 \\ \Big((\boldsymbol{Q}_0\times\boldsymbol{Q}_3)(1-t)^2 \\ \quad +(\boldsymbol{Q}_1\times\boldsymbol{Q}_3)(1-t)t+(\boldsymbol{Q}_2\times\boldsymbol{Q}_3)t^2\Big)\cdot\boldsymbol{P}(t)=0 \end{cases}. \tag{4.5}$$

By making a linear combination of the three equations in Eqn. (4.5),

$$\begin{aligned}&\big((V_{023}(\boldsymbol{Q}_0\times\boldsymbol{Q}_1)-V_{013}(\boldsymbol{Q}_0\times\boldsymbol{Q}_2) \\ &\quad +V_{012}(\boldsymbol{Q}_0\times\boldsymbol{Q}_3))(1-t)^2+(V_{023}(\boldsymbol{Q}_0\times\boldsymbol{Q}_2) \\ &\quad -V_{013}(\boldsymbol{Q}_0\times\boldsymbol{Q}_3+\boldsymbol{Q}_1\times\boldsymbol{Q}_2) \\ &\quad +V_{012}(\boldsymbol{Q}_1\times\boldsymbol{Q}_3))(1-t)t+(V_{023}(\boldsymbol{Q}_0\times\boldsymbol{Q}_3) \\ &-V_{013}(\boldsymbol{Q}_1\times\boldsymbol{Q}_3)+V_{012}(\boldsymbol{Q}_2\times\boldsymbol{Q}_3))t^2\big) \\ &\quad\cdot\boldsymbol{P}(t)=0\end{aligned}, \tag{4.6}$$

where

$$V_{ijk}=\boldsymbol{Q}_i\cdot(\boldsymbol{Q}_j\times\boldsymbol{Q}_k) \tag{4.7}$$

In Eqn. (4.6), the first factor looks a quadratic moving line, however, we can show that it is substantially linear as follows. From

$$\begin{aligned}&(\boldsymbol{Q}_0\times\boldsymbol{Q}_1)\times(\boldsymbol{Q}_2\times\boldsymbol{Q}_3) \\ &\quad=(\boldsymbol{Q}_3\cdot(\boldsymbol{Q}_0\times\boldsymbol{Q}_1))\,\boldsymbol{Q}_2-(\boldsymbol{Q}_2\cdot(\boldsymbol{Q}_0\times\boldsymbol{Q}_1))\,\boldsymbol{Q}_3, \\ &\quad=(\boldsymbol{Q}_1\cdot(\boldsymbol{Q}_2\times\boldsymbol{Q}_3))\,\boldsymbol{Q}_0-(\boldsymbol{Q}_0\cdot(\boldsymbol{Q}_2\times\boldsymbol{Q}_3))\,\boldsymbol{Q}_1\end{aligned}$$

we obtain

$$V_{023}\boldsymbol{Q}_1-V_{013}\boldsymbol{Q}_2+V_{012}\boldsymbol{Q}_3=V_{123}\boldsymbol{Q}_0, \tag{4.8}$$

$$\begin{aligned}&\therefore V_{023}(\boldsymbol{Q}_0\times\boldsymbol{Q}_1)-V_{013}(\boldsymbol{Q}_0\times\boldsymbol{Q}_2)+V_{012}(\boldsymbol{Q}_0\times\boldsymbol{Q}_3) \\ &\quad=V_{123}(\boldsymbol{Q}_0\times\boldsymbol{Q}_0)=\boldsymbol{0}.\end{aligned} \tag{4.9}$$

By substituting Eqn. (4.9) in Eqn. (4.6),

$$\begin{aligned}&\big((V_{023}(\boldsymbol{Q}_0\times\boldsymbol{Q}_2)-V_{013}(\boldsymbol{Q}_0\times\boldsymbol{Q}_3+\boldsymbol{Q}_1\times\boldsymbol{Q}_2) \\ &\quad+V_{012}(\boldsymbol{Q}_1\times\boldsymbol{Q}_3))(1-t)+(V_{023}(\boldsymbol{Q}_0\times\boldsymbol{Q}_3) \\ &\quad-V_{013}(\boldsymbol{Q}_1\times\boldsymbol{Q}_3)+V_{012}(\boldsymbol{Q}_2\times\boldsymbol{Q}_3))\,t\big)\,t \\ &\quad\cdot\boldsymbol{P}(t)=0\end{aligned}. \tag{4.10}$$

Therefore, we have found the linear moving line $\boldsymbol{L}_0(t)$ that follows the Bézier curve $\boldsymbol{P}(t)$:

$$\boldsymbol{L}_0(t)=(1-t)\boldsymbol{L}_{00}+t\boldsymbol{L}_{01}$$

$$\begin{cases}\boldsymbol{L}_{00}=V_{023}(\boldsymbol{Q}_0\times\boldsymbol{Q}_2)-V_{013}(\boldsymbol{Q}_0\times\boldsymbol{Q}_3+\boldsymbol{Q}_1\times\boldsymbol{Q}_2) \\ \quad +V_{012}(\boldsymbol{Q}_1\times\boldsymbol{Q}_3) \\ \boldsymbol{L}_{01}=V_{023}(\boldsymbol{Q}_0\times\boldsymbol{Q}_3)-V_{013}(\boldsymbol{Q}_1\times\boldsymbol{Q}_3) \\ \quad +V_{012}(\boldsymbol{Q}_2\times\boldsymbol{Q}_3)\end{cases} \tag{4.11}$$

The double point $\boldsymbol{D}=w_D\left(x_D,\ \ y_D,\ \ 1\right)$ of the curve $\boldsymbol{P}(t)$ can be obtained as the intersection of two control lines of the linear moving line:

$$\boldsymbol{D}=\boldsymbol{L}_{00}\times\boldsymbol{L}_{01}. \tag{4.12}$$

Fig. 2 shows an example. From four control points of a Bézier curve, we can obtain two control lines $\boldsymbol{L}_{00}$ and $\boldsymbol{L}_{01}$ of the linear moving line, and the double point $\boldsymbol{D}$.

The double point $\boldsymbol{D}$ could be a point at infinity. In next subsections 4.2 and 4.3, we deal with the case that $\boldsymbol{D}$ is not at infinity. In subsection 4.4, we treat the case that $\boldsymbol{D}$ is at infinity.

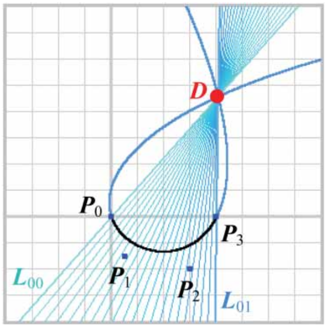

Fig. 2: Linear moving line that follows a cubic Bézier curve.

4.2. Quadratic Parallel Moving Line and Its Existence Area

If the double point $\boldsymbol{D}$ is not at infinity, $\boldsymbol{P}(\infty)$ is not a double point. Thus, by setting $\boldsymbol{K}=\boldsymbol{P}(\infty)$ in Eqn. (4.2), we can derive the quadratic parallel moving line $\boldsymbol{L}_1[t]$ that follows the curve $\boldsymbol{P}(t)$. From Eqn. (4.3),

$$\begin{aligned}
&\lim_{\tau\to\infty}\frac{\boldsymbol{P}(\tau)\times\boldsymbol{P}(t)}{(t-\tau)\tau^2}\\
&=\begin{bmatrix}1 & -1 & 1\end{bmatrix}\\
&\times\begin{bmatrix}\boldsymbol{Q}_0\times\boldsymbol{Q}_1 & \boldsymbol{Q}_0\times\boldsymbol{Q}_2 & \boldsymbol{Q}_0\times\boldsymbol{Q}_3\\ \boldsymbol{Q}_0\times\boldsymbol{Q}_2 & \boldsymbol{Q}_0\times\boldsymbol{Q}_3+\boldsymbol{Q}_1\times\boldsymbol{Q}_2 & \boldsymbol{Q}_1\times\boldsymbol{Q}_3\\ \boldsymbol{Q}_0\times\boldsymbol{Q}_3 & \boldsymbol{Q}_1\times\boldsymbol{Q}_3 & \boldsymbol{Q}_2\times\boldsymbol{Q}_3\end{bmatrix}\\
&\times\begin{bmatrix}(1-t)^2\\ (1-t)t\\ t^2\end{bmatrix}\\
&=\left(-\boldsymbol{Q}_0\times\boldsymbol{Q}_1+\boldsymbol{Q}_0\times\boldsymbol{Q}_2-\boldsymbol{Q}_0\times\boldsymbol{Q}_3\right)(1-t)^2\\
&\quad+\left(-\boldsymbol{Q}_0\times\boldsymbol{Q}_2+\boldsymbol{Q}_0\times\boldsymbol{Q}_3+\boldsymbol{Q}_1\times\boldsymbol{Q}_2-\boldsymbol{Q}_1\times\boldsymbol{Q}_3\right)\\
&\quad\times(1-t)t+\left(-\boldsymbol{Q}_0\times\boldsymbol{Q}_3+\boldsymbol{Q}_1\times\boldsymbol{Q}_3-\boldsymbol{Q}_2\times\boldsymbol{Q}_3\right)t^2
\end{aligned}\quad. \tag{4.13}$$

Thus, the quadratic parallel moving line $\boldsymbol{L}_1(t)$ is:

$$\begin{aligned}
&\boldsymbol{L}_1(t)=(1-t)^2\boldsymbol{L}_{10}+2(1-t)t\boldsymbol{L}_{11}+t^2\boldsymbol{L}_{12}\\
&\begin{cases}\boldsymbol{L}_{10}=\left(-\boldsymbol{Q}_0\times\boldsymbol{Q}_1+\boldsymbol{Q}_0\times\boldsymbol{Q}_2-\boldsymbol{Q}_0\times\boldsymbol{Q}_3\right)\\ \boldsymbol{L}_{11}=\frac{1}{2}\left(-\boldsymbol{Q}_0\times\boldsymbol{Q}_2+\boldsymbol{Q}_0\times\boldsymbol{Q}_3+\boldsymbol{Q}_1\times\boldsymbol{Q}_2\right.\\ \quad\left.-\boldsymbol{Q}_1\times\boldsymbol{Q}_3\right)\\ \boldsymbol{L}_{12}=\left(-\boldsymbol{Q}_0\times\boldsymbol{Q}_3+\boldsymbol{Q}_1\times\boldsymbol{Q}_3-\boldsymbol{Q}_2\times\boldsymbol{Q}_3\right)\end{cases}.
\end{aligned} \tag{4.14}$$

Here, the differences of adjacent control line pairs can be expressed:

$$\begin{cases}\boldsymbol{L}_{11}-\boldsymbol{L}_{10}=\frac{1}{2}\left(\boldsymbol{Q}_1-3\,\boldsymbol{Q}_0\right)\times\left(\boldsymbol{Q}_3-\boldsymbol{Q}_2+\boldsymbol{Q}_1-\boldsymbol{Q}_0\right)\\ \quad=\frac{3}{2}\left(\boldsymbol{P}_1-\boldsymbol{P}_0\right)\times\left(\boldsymbol{P}_3-3\,\boldsymbol{P}_2+3\,\boldsymbol{P}_1-\boldsymbol{P}_0\right)\\ \boldsymbol{L}_{12}-\boldsymbol{L}_{11}=\frac{1}{2}\left(3\boldsymbol{Q}_3-\boldsymbol{Q}_2\right)\times\left(\boldsymbol{Q}_3-\boldsymbol{Q}_2+\boldsymbol{Q}_1-\boldsymbol{Q}_0\right)\\ \quad=\frac{3}{2}\left(\boldsymbol{P}_3-\boldsymbol{P}_2\right)\times\left(\boldsymbol{P}_3-3\,\boldsymbol{P}_2+3\,\boldsymbol{P}_1-\boldsymbol{P}_0\right)\end{cases}. \tag{4.15}$$

Since $\boldsymbol{P}(t)$ is not rational, weight of each control point $\boldsymbol{P}_i$ is 1. Therefore, each factor in Eqn. (4.15) can be described as follows:

$$\begin{cases}\boldsymbol{P}_3-3\boldsymbol{P}_2+3\boldsymbol{P}_1-\boldsymbol{P}_0=\left(x_{d0123},\quad y_{d0123},\quad 0\right)\\ \boldsymbol{P}_1-\boldsymbol{P}_0=\left(x_{d01},\quad y_{d01},\quad 0\right)\\ \boldsymbol{P}_3-\boldsymbol{P}_2=\left(x_{d23},\quad y_{d23},\quad 0\right)\end{cases}. \tag{4.16}$$

$$\therefore\begin{cases}\boldsymbol{L}_{11}-\boldsymbol{L}_{10}=\left(0,\ 0,\ \frac{3}{2}\left(x_{d01}y_{d0123}-y_{d01}x_{d0123}\right)\right)\\ \boldsymbol{L}_{12}-\boldsymbol{L}_{11}=\left(0,\ 0,\ \frac{3}{2}\left(x_{d23}y_{d0123}-y_{d23}x_{d0123}\right)\right)\end{cases}. \tag{4.17}$$

From Eqn. (4.17) each control line of $\boldsymbol{L}_1(t)$ can be expressed:

$$\begin{cases}\boldsymbol{L}_{10}=\left(a_1,\quad b_1,\quad c_{10}\right)\\ \boldsymbol{L}_{11}=\left(a_1,\quad b_1,\quad c_{11}\right)\\ \boldsymbol{L}_{12}=\left(a_1,\quad b_1,\quad c_{12}\right)\end{cases}, \tag{4.18}$$

$$\begin{aligned}&\therefore\boldsymbol{L}_1(t)=\left(a_1,\quad b_1,\quad c_1(t)\right)\\ &c_1(t)=(1-t)^2c_{10}+2(1-t)tc_{11}+t^2c_{12}\end{aligned}, \tag{4.19}$$

where a_1 and b_1 are constants. This means the motion of $\boldsymbol{L}_1(t)$ is a parallel translation of a line, and quadratic function $c_1(t)$ gives the position. Therefore, $\boldsymbol{L}_1(t)$ exists in a certain half plane π_E; as t increases, $\boldsymbol{L}_1(t)$ approaches to the border of π_E, reaches the border at $t=t_E$, then returns to the beginning side. The parameter value t_E is obtained by:

$$\dot{\boldsymbol{L}}_1(t_E)=\boldsymbol{0}, \tag{4.20}$$

$$\therefore t_E=\frac{c_{10}-c_{11}}{c_{10}-2c_{11}+c_{12}}. \tag{4.21}$$

The curve $\boldsymbol{P}(t)$ is tangent to the border line $\boldsymbol{L}_1(t_E)$, and the tangent point $\boldsymbol{E}=w_E\left(x_E,\quad y_E,\quad 1\right)$ is obtained as:

$$\boldsymbol{E}=\boldsymbol{L}_0(t_E)\times\boldsymbol{L}_1(t_E). \tag{4.22}$$

Fig. 3 shows an example. From four control points of a Bézier curve, we can obtain three control lines $\boldsymbol{L}_{10},\boldsymbol{L}_{11}$ and $\boldsymbol{L}_{12}$ of the quadratic parallel moving line, the borderline $\boldsymbol{L}_1(t_E)$ of the half plane π_E, and the tangent point $\boldsymbol{E}$.

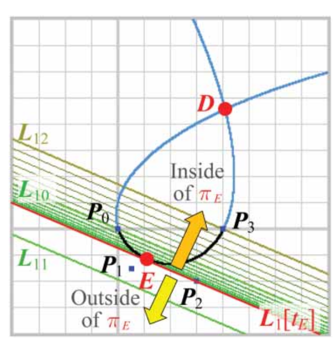

Fig. 3: Quadratic parallel moving line that follows a cubic Bézier curve.

4.3. Classification, Reference Points, and Affine Transformation from Primitive Cubic

The feature of the double point $\boldsymbol{D}$ depends on its relationship to the half plane π_E, *i.e.* whether $\boldsymbol{D}$ is inside, outside, or on the border of π_E. On the curve $\boldsymbol{P}(t)$, the parameter value t at the double point $\boldsymbol{D}$ can be obtained by a quadratic equation:

$$\boldsymbol{L}_1(t) \cdot \boldsymbol{D} = 0. \tag{4.23}$$

Let D be the discriminant of Eqn. (4.23). Then the sign of D corresponds to the relationship between $\boldsymbol{D}$ and π_E, from which the curve $\boldsymbol{P}(t)$ can be classified as follows:

- $D > 0$: Case 1. Crunode,
- $D = 0$: Case 2. Cusp,
- $D < 0$: Case 3. Acnode.

[Case 1. Crunode]

If $D > 0$, Eqn. (4.23) has two real roots, which means $\boldsymbol{D}$ is a crunode (self intersection) because the curve $\boldsymbol{P}(t)$ go through the double point $\boldsymbol{D}$ twice. Let t_F be the greater root, then $\boldsymbol{L}_0(t_F)$ is tangent to the curve at $\boldsymbol{D}$. Let $\boldsymbol{F} = w_F\,(x_F,\ \ y_F,\ \ 1)$ be the intersection of two lines $\boldsymbol{L}_0(t_F)$ and $\boldsymbol{L}_1(t_E)$:

$$\boldsymbol{F} = \boldsymbol{L}_0(t_F) \times \boldsymbol{L}_1(t_E), \tag{4.24}$$

Next, we make a linear transformation of t:

$$\hat{t} = \frac{t - t_E}{t_F - t_E}, \tag{4.25}$$

and let $\hat{\boldsymbol{L}}_0(\hat{t})$ and $\hat{\boldsymbol{L}}_1(\hat{t})$ be equivalent moving lines to $\boldsymbol{L}_0(t)$ and $\boldsymbol{L}_1(t)$, respectively. Here, $\hat{\boldsymbol{L}}_0(0)$ is equivalent to $\boldsymbol{L}_0(t_E)$ which contains $\boldsymbol{D}$ and $\boldsymbol{E}$, $\hat{\boldsymbol{L}}_0(1)$ is equivalent to $\boldsymbol{L}_0(t_F)$ which contains $\boldsymbol{D}$ and $\boldsymbol{F}$, and $\hat{\boldsymbol{L}}_0(\infty)$ is equivalent to $\boldsymbol{L}_0(\infty)$ which contains $\boldsymbol{D}$ and $\boldsymbol{P}(\infty)$. Therefore, $\hat{\boldsymbol{L}}_0(\hat{t})$ can be described as:

$$\begin{gathered}\hat{\boldsymbol{L}}_0(\hat{t}) = (1 - \hat{t})\hat{\boldsymbol{L}}_{00} + \hat{t}\hat{\boldsymbol{L}}_{01} \\ \begin{cases}\hat{\boldsymbol{L}}_{00} = k_{L0}\left(\hat{\boldsymbol{D}} \times \hat{\boldsymbol{E}}\right) \\ \hat{\boldsymbol{L}}_{01} = k_{L0}\left(\hat{\boldsymbol{D}} \times \hat{\boldsymbol{F}}\right)\end{cases},\end{gathered} \tag{4.26}$$

where $\hat{\boldsymbol{D}} = \boldsymbol{D}/w_D = (x_D,\ \ y_D,\ \ 1)$, $\hat{\boldsymbol{E}} = \boldsymbol{E}/w_E = (x_E,\ \ y_E,\ \ 1)$, $\hat{\boldsymbol{F}} = \boldsymbol{F}/w_F = (x_F,\ \ y_F,\ \ 1)$, and k_{L0} is a constant. Likewise, $\hat{\boldsymbol{L}}_1(0)$ is equivalent to $\boldsymbol{L}_1[t_E]$ which contains $\boldsymbol{P}(\infty)$ and $\boldsymbol{E}$, $\hat{\boldsymbol{L}}_1(1)$ is equivalent to $\boldsymbol{L}_1(t_F)$ which contains $\boldsymbol{P}(\infty)$ and $\boldsymbol{F}$, $\hat{\boldsymbol{L}}_1(\infty)$ is equivalent to $\boldsymbol{L}_1(\infty)$ which is the set of points at infinity, and $\dot{\hat{\boldsymbol{L}}}_1(0) = \boldsymbol{0}$. Therefore, $\hat{\boldsymbol{L}}_1(\hat{t})$ can be described as:

$$\begin{gathered}\hat{\boldsymbol{L}}_1(\hat{t}) = (1 - \hat{t})^2\hat{\boldsymbol{L}}_{10} + 2(1 - \hat{t})\hat{t}\hat{\boldsymbol{L}}_{11} + \hat{t}^2\hat{\boldsymbol{L}}_{12} \\ \begin{cases}\hat{\boldsymbol{L}}_{10} = k_{L1}\left(\boldsymbol{P}(\infty) \times \hat{\boldsymbol{E}}\right) \\ \hat{\boldsymbol{L}}_{11} = k_{L1}\left(\boldsymbol{P}(\infty) \times \hat{\boldsymbol{E}}\right) \\ \hat{\boldsymbol{L}}_{12} = k_{L1}\left(\boldsymbol{P}(\infty) \times \hat{\boldsymbol{D}}\right)\end{cases}.\end{gathered} \tag{4.27}$$

where k_{L1} is a constant. The intersection of these moving lines:

$$\hat{\boldsymbol{P}}(\hat{t}) = \hat{\boldsymbol{L}}_0(\hat{t}) \times \hat{\boldsymbol{L}}_1(\hat{t}) \tag{4.28}$$

is equivalent to the curve $\boldsymbol{P}(t)$.

This curve $\hat{\boldsymbol{P}}(\hat{t})$ can be obtained by an affine transformation of the following primitive cubic:

$$\tilde{\boldsymbol{P}}_{C1}(\tilde{t}) = \left(\tilde{t}^2 - 1,\ \ \frac{1}{\sqrt{3}}\left(\tilde{t}^3 - \tilde{t}\right),\ \ 1\right) \tag{4.29}$$

$$\text{or } x^3 + x^2 - 3y^2 = 0 \tag{4.30}$$

because this curve $\tilde{\boldsymbol{P}}_{C1}(\tilde{t})$ can also be defined from the following reference points and moving lines:

$$\begin{gathered}\tilde{\boldsymbol{D}} = (0,\ \ 0,\ \ 1),\quad \tilde{\boldsymbol{E}} = (-1,\ \ 0,\ \ 1), \\ \tilde{\boldsymbol{F}} = \left(-1,\ \ -\frac{1}{\sqrt{3}},\ \ 1\right),\quad \tilde{\boldsymbol{P}}_\infty = (0,\ \ 1,\ \ 0),\end{gathered} \tag{4.31}$$

$$\begin{gathered}\tilde{\boldsymbol{L}}_0(\tilde{t}) = (1 - \tilde{t})\tilde{\boldsymbol{L}}_{00} + \tilde{t}\tilde{\boldsymbol{L}}_{01} \\ \begin{cases}\tilde{\boldsymbol{L}}_{00} = \tilde{\boldsymbol{D}} \times \tilde{\boldsymbol{E}} = \left(0,\ \ -1,\ \ 0\right) \\ \tilde{\boldsymbol{L}}_{01} = \tilde{\boldsymbol{D}} \times \tilde{\boldsymbol{F}} = \left(\frac{1}{\sqrt{3}},\ \ -1,\ \ 0\right)\end{cases},\end{gathered} \tag{4.32}$$

$$\begin{gathered}\tilde{\boldsymbol{L}}_1(\tilde{t}) = (1 - \tilde{t})^2\tilde{\boldsymbol{L}}_{10} + 2\,(1 - \tilde{t})\tilde{t}\tilde{\boldsymbol{L}}_{11} + \tilde{t}^2\tilde{\boldsymbol{L}}_{12} \\ \begin{cases}\tilde{\boldsymbol{L}}_{10} = \tilde{\boldsymbol{P}}_\infty \times \tilde{\boldsymbol{E}} = \left(1,\ \ 0,\ \ 1\right) \\ \tilde{\boldsymbol{L}}_{11} = \tilde{\boldsymbol{P}}_\infty \times \tilde{\boldsymbol{E}} = \left(1,\ \ 0,\ \ 1\right) \\ \tilde{\boldsymbol{L}}_{12} = \tilde{\boldsymbol{P}}_\infty \times \tilde{\boldsymbol{D}} = \left(1,\ \ 0,\ \ 0\right)\end{cases},\end{gathered} \tag{4.33}$$

$$\tilde{\boldsymbol{P}}(\tilde{t}) = \tilde{\boldsymbol{L}}_0(\tilde{t}) \times \tilde{\boldsymbol{L}}_1(\tilde{t}). \tag{4.34}$$

The affine transformation matrix $\mathbf{M}_{C1}$ that maps $\tilde{\boldsymbol{P}}_{C1}(\tilde{t})$ into $\boldsymbol{P}(t)$ can be obtained so as to map three reference points $\tilde{\boldsymbol{D}}$, $\tilde{\boldsymbol{E}}$ and $\tilde{\boldsymbol{F}}$ into $\hat{\boldsymbol{D}}$, $\hat{\boldsymbol{E}}$ and $\hat{\boldsymbol{F}}$, respectively:

$$\begin{bmatrix}\tilde{\boldsymbol{D}} \\ \tilde{\boldsymbol{E}} \\ \tilde{\boldsymbol{F}}\end{bmatrix}\mathbf{M}_{C1} = \begin{bmatrix}\hat{\boldsymbol{D}} \\ \hat{\boldsymbol{E}} \\ \hat{\boldsymbol{F}}\end{bmatrix} \tag{4.35}$$

$$\therefore \mathbf{M}_{C1} = \begin{pmatrix} x_D - x_E & y_D - y_E & 0 \\ \sqrt{3}\,(x_E - x_F) & \sqrt{3}\,(y_E - y_F) & 0 \\ x_D & y_D & 1\end{pmatrix} \tag{4.36}$$

and the parameter transformation is:

$$t = \left(1 - \tilde{t}\right)t_E + \tilde{t}t_F. \tag{4.37}$$

Fig. 4 shows an example Bézier curve with a crunode and its correspondence to the primitive cubic.

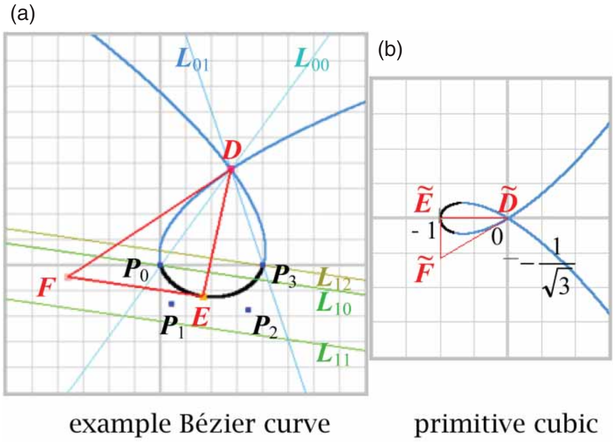

Fig. 4: Case 1. Crunode.

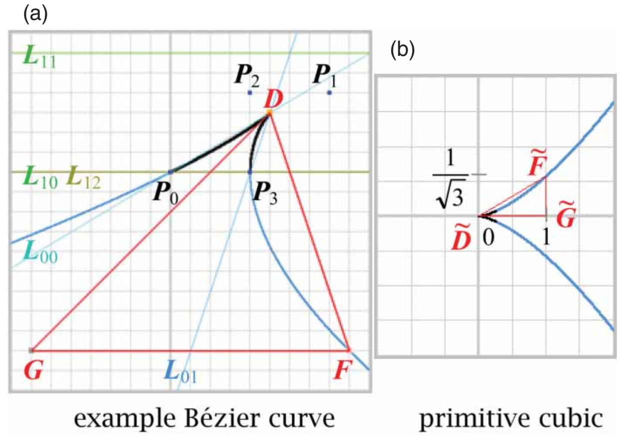

Fig. 5: Case 2. Cusp.

Here, the whole shape of each curve is drawn in blue, and parameter interval $0 \le t \le 1$ of original Bézier is drawn in black.

[Case 2. Cusp]

If $D = 0$, Eqn. (4.23) has double roots, which means $\boldsymbol{D}$ is a cusp because the curve $\boldsymbol{P}(t)$ go through the double point $\boldsymbol{D}$ once. Let t_F be

$$t_F = t_E + 1. \tag{4.38}$$

Let $\boldsymbol{G} = w_G\,(x_G,\ \ y_G,\ \ 1)$ be the intersection of two lines $\boldsymbol{L}_0(t_E)$ and $\boldsymbol{L}_1(t_F)$, and let $\boldsymbol{F} = w_F\,(x_F,\ \ y_F,\ \ 1)$ be the intersection of two lines $\boldsymbol{L}_0(t_F)$ and $\boldsymbol{L}_1(t_F)$:

$$\begin{cases} \boldsymbol{G} = \boldsymbol{L}_0(t_E) \times \boldsymbol{L}_1(t_F) \\ \boldsymbol{F} = \boldsymbol{L}_0(t_F) \times \boldsymbol{L}_1(t_F) \end{cases}. \tag{4.39}$$

In the same way as Case 1, the curve $\boldsymbol{P}(t)$ can be obtained by an affine transformation and parameter transformation of the following primitive cubic:

$$\tilde{\boldsymbol{P}}_{C2}(\tilde{t}) = \left(\tilde{t}^2,\ \ \tfrac{1}{\sqrt{3}}\tilde{t}^3,\ \ 1\right) \tag{4.40}$$

$$\text{or } x^3 - 3y^2 = 0. \tag{4.41}$$

By using the correspondence of three reference points $\boldsymbol{D}$, $\boldsymbol{G}$ and $\boldsymbol{F}$, the affine transformation matrix $\mathbf{M}_{C2}$ that maps $\tilde{\boldsymbol{P}}_{C2}(\tilde{t})$ into $\boldsymbol{P}(t)$ can be obtained as follows:

$$\begin{bmatrix} \tilde{\boldsymbol{D}} \\ \tilde{\boldsymbol{G}} \\ \tilde{\boldsymbol{F}} \end{bmatrix} \mathbf{M}_{C2} = \begin{bmatrix} \hat{\boldsymbol{D}} \\ \hat{\boldsymbol{D}} \\ \hat{\boldsymbol{D}} \end{bmatrix} \tag{4.42}$$

where $\hat{\boldsymbol{D}} = \boldsymbol{D}/w_D = (x_D,\ \ y_D,\ \ 1)$, $\hat{\boldsymbol{G}} = \boldsymbol{G}/w_G = (x_G,\ \ y_G,\ \ 1)$, $\hat{\boldsymbol{F}} = \boldsymbol{F}/w_F = (x_F,\ \ y_F,\ \ 1)$, and

$$\tilde{\boldsymbol{D}} = (0,\ 0,\ 1),\quad \tilde{\boldsymbol{G}} = (1,\ 0,\ 1),\quad \tilde{\boldsymbol{F}} = \left(1,\ \tfrac{1}{\sqrt{3}},\ 1\right), \tag{4.43}$$

$$\therefore \mathbf{M}_{C2} = \begin{pmatrix} x_G - x_D & y_G - y_D & 0 \\ \sqrt{3}\,(x_F - x_G) & \sqrt{3}\,(y_F - y_G) & 0 \\ x_D & y_D & 1 \end{pmatrix}, \tag{4.44}$$

and the parameter transformation is:

$$t = \tilde{t} + t_E. \tag{4.45}$$

Fig. 5 shows an example Bézier curve with a cusp and its correspondence to the primitive cubic.

[Case 3. Acnode]

If $D < 0$, Eqn. (4.23) has complex roots, which means $\boldsymbol{D}$ is an acnode (isolate point) because the curve $\boldsymbol{P}(t)$ does not go through the double point $\boldsymbol{D}$. In this case, let $\boldsymbol{G} = \hat{\boldsymbol{G}} = (x_G,\ \ y_G,\ \ 1)$ be

$$\hat{\boldsymbol{G}} = 2\,\hat{\boldsymbol{E}} - \hat{\boldsymbol{D}}, \tag{4.46}$$

where $\hat{\boldsymbol{D}} = \boldsymbol{D}/w_D = (x_D,\ \ y_D,\ \ 1)$ and $\hat{\boldsymbol{E}} = \boldsymbol{E}/w_E = (x_E,\ \ y_E,\ \ 1)$. Since $\boldsymbol{E}$ is the midpoint of line segment $\boldsymbol{DG}$, the point $\boldsymbol{G}$ is inside of the half plane π_E and the following quadratic equation has two real roots:

$$\boldsymbol{L}_1(t) \cdot \boldsymbol{G} = 0. \tag{4.47}$$

Let t_F be the greater root of Eqn. (4.23), and let $F = w_F\,(x_F,\ \ y_F,\ \ 1)$ be the intersection of two lines $\boldsymbol{L}_0(t_F)$ and $\boldsymbol{L}_1(t_F)$:

$$\boldsymbol{F} = \boldsymbol{L}_0(t_F) \times \boldsymbol{L}_1(t_F). \tag{4.48}$$

In the same way as Case 1, the curve $\boldsymbol{P}(t)$ can be obtained by an affine transformation and parameter transformation of the following primitive cubic:

$$\tilde{\boldsymbol{P}}_{C3}\,(\tilde{t}) = \left(\tilde{t}^2 + 1,\ \ \tfrac{1}{\sqrt{3}}\left(\tilde{t}^3 + \tilde{t}\right),\ \ 1\right) \tag{4.49}$$

$$\text{or } x^3 - x^2 - 3y^2 = 0. \tag{4.50}$$

By using three reference points $\boldsymbol{D}$, $\boldsymbol{G}$ and $\boldsymbol{F}$, the affine transformation matrix $\mathbf{M}_{C3}$ that maps $\tilde{\boldsymbol{P}}_{C3}(\tilde{t})$ into $\boldsymbol{P}(t)$

can be obtained as follows:

$$\begin{bmatrix}\tilde{\boldsymbol{D}}\\ \tilde{\boldsymbol{G}}\\ \tilde{\boldsymbol{F}}\end{bmatrix}\mathbf{M}_{C3}=\begin{bmatrix}\hat{\boldsymbol{D}}\\ \hat{\boldsymbol{G}}\\ \hat{\boldsymbol{F}}\end{bmatrix} \tag{4.51}$$

where $\hat{\boldsymbol{F}}=\boldsymbol{F}/w_F=(x_F,\ \ y_F,\ \ 1)$, and

$$\tilde{\boldsymbol{D}}=(0,\ 0,\ 1),\quad \tilde{\boldsymbol{G}}=(2,\ 0,\ 1),\quad \tilde{\boldsymbol{F}}=\left(2,\ \frac{2}{\sqrt{3}},\ 1\right), \tag{4.52}$$

$$\therefore \mathbf{M}_{C3}=\begin{pmatrix}\frac{1}{2}(x_G-x_D) & \frac{1}{2}(y_G-y_D) & 0\\ \frac{\sqrt{3}}{2}(x_F-x_G) & \frac{\sqrt{3}}{2}(y_F-y_G) & 0\\ x_D & y_D & 1\end{pmatrix}, \tag{4.53}$$

and the parameter transformation is:

$$t=\left(1-\tilde{t}\right)t_E+\tilde{t}\,t_F. \tag{4.54}$$

Fig. 6 shows an example Bézier curve with a cusp and its correspondence to the primitive cubic.

4.4. Double Point at Infinity

If the double point $\boldsymbol{D}$ is at infinity, the linear moving line $\boldsymbol{L}_0(t)$ contains $\boldsymbol{P}(\infty)$ and is equivalent to $\boldsymbol{L}_1(t)$. Thus, we need a quadratic moving line with another fixed point. Here, we use the inflection point. Inflection points of curve $\boldsymbol{P}(t)$ can be obtained by the following equation [9]:

$$\dot{\boldsymbol{P}}(t)\times\ddot{\boldsymbol{P}}(t)=\mathbf{0}, \tag{4.55}$$

$$\therefore (1-t)^2\left(\boldsymbol{P}_1-\boldsymbol{P}_0\right)\times\left(\boldsymbol{P}_2-\boldsymbol{P}_1\right)+t(1-t)\left(\boldsymbol{P}_1-\boldsymbol{P}_0\right)\times\left(\boldsymbol{P}_3-\boldsymbol{P}_2\right)+t^2\left(\boldsymbol{P}_2-\boldsymbol{P}_1\right)\times\left(\boldsymbol{P}_3-\boldsymbol{P}_2\right)=\mathbf{0}, \tag{4.56}$$

Since the parallel moving line $\boldsymbol{L}_1(t)$ is linear,

$$\boldsymbol{L}_{11}-\boldsymbol{L}_{10}=\boldsymbol{L}_{12}-\boldsymbol{L}_{11}. \tag{4.57}$$

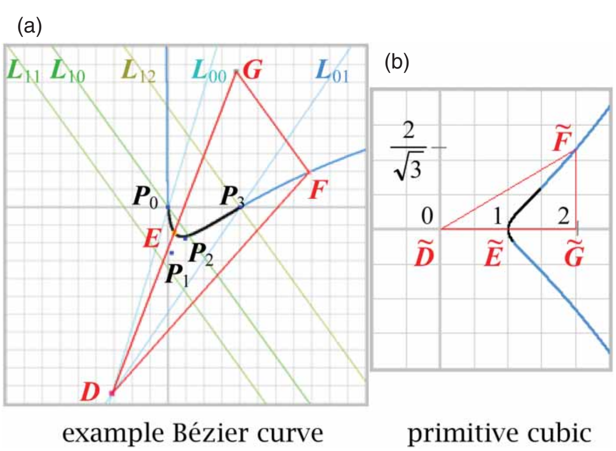

Fig. 6: Case 3. Acnode.

From Eqn. (4.14) and (4.47),

$$\begin{aligned}&\left(\boldsymbol{P}_1-\boldsymbol{P}_0\right)\times\left(\left(\boldsymbol{P}_1-\boldsymbol{P}_0\right)-2\left(\boldsymbol{P}_2-\boldsymbol{P}_1\right)+\left(\boldsymbol{P}_3-\boldsymbol{P}_2\right)\right)\\ &\quad=\left(\boldsymbol{P}_3-\boldsymbol{P}_2\right)\times\left(\left(\boldsymbol{P}_1-\boldsymbol{P}_0\right)-2\left(\boldsymbol{P}_2-\boldsymbol{P}_1\right)+\left(\boldsymbol{P}_3-\boldsymbol{P}_2\right)\right),\end{aligned} \tag{4.58}$$

$$\begin{aligned}\therefore &\left(\boldsymbol{P}_1-\boldsymbol{P}_0\right)\times\left(\boldsymbol{P}_3-\boldsymbol{P}_2\right)=\left(\boldsymbol{P}_1-\boldsymbol{P}_0\right)\times\left(\boldsymbol{P}_2-\boldsymbol{P}_1\right)\\ &\quad+\left(\boldsymbol{P}_2-\boldsymbol{P}_1\right)\times\left(\boldsymbol{P}_3-\boldsymbol{P}_2\right).\end{aligned} \tag{4.59}$$

By substituting Eqn. (4.59) into Eqn. (4.56),

$$\begin{aligned}&(1-t)\left(\boldsymbol{P}_1-\boldsymbol{P}_0\right)\times\left(\boldsymbol{P}_2-\boldsymbol{P}_1\right)\\ &\quad+t\left(\boldsymbol{P}_2-\boldsymbol{P}_1\right)\times\left(\boldsymbol{P}_3-\boldsymbol{P}_2\right)=\mathbf{0}.\end{aligned} \tag{4.60}$$

Let t_I be the root of Eqn. (4.60), then $\boldsymbol{I}=(x_I,\ \ y_I,\ \ 1)=\boldsymbol{P}(t_I)$ is the inflection point of $\boldsymbol{P}(t)$. By substituting $\tau=t_I$ into Eqn. (4.3), the quadratic moving line $\boldsymbol{L}_2(t)$ that contains the fixed point $\boldsymbol{I}$ and follows the curve $\boldsymbol{P}(t)$ is:

$$\boldsymbol{L}_2(t)=(1-t)^2\boldsymbol{L}_{20}+2(1-t)t\boldsymbol{L}_{21}+t^2\boldsymbol{L}_{22}$$
$$\begin{cases}\boldsymbol{L}_{20}=(1-t_I)^2\left(\boldsymbol{Q}_0\times\boldsymbol{Q}_1\right)+(1-t_I)t_I\left(\boldsymbol{Q}_0\times\boldsymbol{Q}_2\right)\\ \quad+t_I^2\left(\boldsymbol{Q}_0\times\boldsymbol{Q}_3\right)\\ \boldsymbol{L}_{21}=\frac{1}{2}\left((1-t_I)^2\left(\boldsymbol{Q}_0\times\boldsymbol{Q}_2\right)\right.\\ \quad\left.+(1-t_I)t_I\left(\boldsymbol{Q}_0\times\boldsymbol{Q}_3+\boldsymbol{Q}_1\times\boldsymbol{Q}_2\right)+t_I^2\left(\boldsymbol{Q}_1\times\boldsymbol{Q}_3\right)\right)\\ \boldsymbol{L}_{22}=(1-t_I)^2\left(\boldsymbol{Q}_0\times\boldsymbol{Q}_3\right)+(1-t_I)t_I\left(\boldsymbol{Q}_1\times\boldsymbol{Q}_3\right)\\ \quad+t_I^2\left(\boldsymbol{Q}_2\times\boldsymbol{Q}_3\right)\end{cases} \tag{4.61}$$

[Case 4. Explicit Cubic]

Let t_F be

$$t_F=t_I+1. \tag{4.62}$$

Let $\boldsymbol{G}=w_G(x_G,\ \ y_G,\ \ 1)$ be the intersection of two lines $\boldsymbol{L}_0(t_F)$ and $\boldsymbol{L}_2(t_I)$, and let $\boldsymbol{F}=w_F(x_F,\ \ y_F,\ \ 1)$ be the intersection of two lines $\boldsymbol{L}_0(t_F)$ and $\boldsymbol{L}_2(t_F)$:

$$\begin{cases}\boldsymbol{G}=\boldsymbol{L}_0(t_F)\times\boldsymbol{L}_2(t_I)\\ \boldsymbol{F}=\boldsymbol{L}_0(t_F)\times\boldsymbol{L}_2(t_F)\end{cases}. \tag{4.63}$$

In the same way as Case 1, the curve $\boldsymbol{P}(t)$ can be obtained by an affine transformation and parameter transformation of the following primitive cubic:

$$\tilde{\boldsymbol{P}}_{C4}[\tilde{t}]=(\tilde{t},\ \ \tilde{t}^3,\ \ 1) \tag{4.64}$$

$$\text{or } x^3-y=0. \tag{4.65}$$

By using the correspondence of three reference points $\boldsymbol{I}$, $\boldsymbol{G}$ and $\boldsymbol{F}$, the affine transformation matrix $\mathbf{M}_{C4}$ that

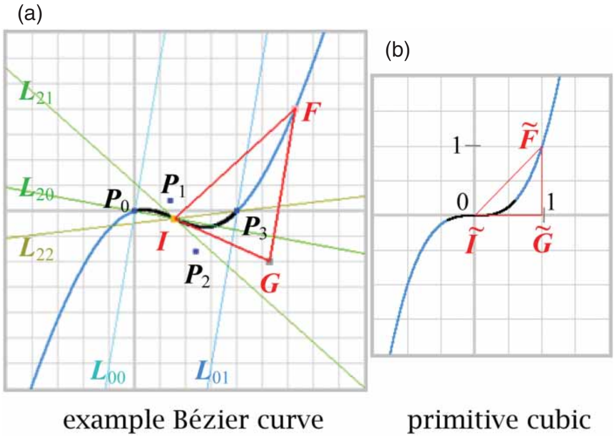

Fig. 7: Case 4. Explicit cubic.

maps $\tilde{\boldsymbol{P}}_{C4}(\tilde{t})$ into $\boldsymbol{P}(t)$ can be obtained as follows:

$$\begin{bmatrix} \tilde{\boldsymbol{I}} \\ \tilde{\boldsymbol{G}} \\ \tilde{\boldsymbol{F}} \end{bmatrix} \mathbf{M}_{C4} = \begin{bmatrix} \hat{\boldsymbol{I}} \\ \hat{\boldsymbol{G}} \\ \hat{\boldsymbol{F}} \end{bmatrix} \tag{4.66}$$

where $\hat{\boldsymbol{I}} = \boldsymbol{I} = (x_I,\ y_I,\ 1)$, $\hat{\boldsymbol{G}} = \boldsymbol{G}/w_G = (x_G,\ \ y_G,\ \ 1)$, $\hat{\boldsymbol{F}} = \boldsymbol{F}/w_F = (x_F,\ \ y_F,\ \ 1)$, and

$$\tilde{\boldsymbol{D}} = (0,\ \ 0,\ \ 1)\,,\quad \tilde{\boldsymbol{G}} = (1,\ \ 0,\ \ 1)\,,\quad \tilde{\boldsymbol{F}} = (1,\ \ 1,\ \ 1)\,, \tag{4.67}$$

$$\therefore \mathbf{M}_{C4} = \begin{pmatrix} x_G - x_I & y_G - y_I & 0 \\ x_F - x_G & y_F - y_G & 0 \\ x_I & y_I & 1 \end{pmatrix}, \tag{4.68}$$

and the parameter transformation is:

$$t = \tilde{t} + t_I. \tag{4.69}$$

Fig. 7 shows an example Bézier curve with a double point at infinity and its correspondence to the primitive cubic.

5. DISCUSSION

In this section, we discuss some supplementary subjects on the theory and implementation.

5.1. Completeness of Classification

In this subsection, we confirm that there is no other possibility than the above four cases.

In Eqn. (4.11), $\boldsymbol{L}_0(t)$ can always be calculated, and Eqn. (4.10) shows that the result $\boldsymbol{L}_0(t)$ must satisfy

$$\boldsymbol{L}_0(t) \cdot \boldsymbol{P}(t) = 0. \tag{4.70}$$

If $\boldsymbol{L}_0(t)$ is not a linear moving line, it must satisfy

$$\boldsymbol{L}_0(t) = \mathbf{0} \quad (\text{for } \forall t). \tag{4.71}$$

Suppose Eqn. (4.71) is true,

$$\boldsymbol{L}_{01} = V_{023}(\boldsymbol{Q}_0 \times \boldsymbol{Q}_3) - V_{013}(\boldsymbol{Q}_1 \times \boldsymbol{Q}_3) + V_{012}(\boldsymbol{Q}_2 \times \boldsymbol{Q}_3) = \mathbf{0} \tag{4.72}$$

$$\therefore (V_{023}\boldsymbol{Q}_0 - V_{013}\boldsymbol{Q}_1 + V_{012}\boldsymbol{Q}_2) \times \boldsymbol{Q}_3 = \mathbf{0}. \tag{4.73}$$

From Eqn. (4.8),

$$\therefore V_{123}\boldsymbol{Q}_0 - V_{023}\boldsymbol{Q}_1 + V_{013}\boldsymbol{Q}_2 = V_{012}\boldsymbol{Q}_3 \quad , \tag{4.74}$$

$$\therefore\ V_{123}(\boldsymbol{Q}_0 \times \boldsymbol{Q}_3) - V_{023}(\boldsymbol{Q}_1 \times \boldsymbol{Q}_3) + V_{013}(\boldsymbol{Q}_2 \times \boldsymbol{Q}_3) = V_{012}(\boldsymbol{Q}_3 \times \boldsymbol{Q}_3) = \mathbf{0},$$

$$(V_{123}\boldsymbol{Q}_0 - V_{023}\boldsymbol{Q}_1 + V_{013}\boldsymbol{Q}_2) \times \boldsymbol{Q}_3 = \mathbf{0}. \tag{4.75}$$

From Eqn. (4.73) and (4.75), two constants k_0 and k_1 exist such that

$$\begin{cases} V_{023}\boldsymbol{Q}_0 - V_{013}\boldsymbol{Q}_1 + V_{012}\boldsymbol{Q}_2 = k_0\boldsymbol{Q}_3 \\ V_{123}\boldsymbol{Q}_0 - V_{023}\boldsymbol{Q}_1 + V_{013}\boldsymbol{Q}_2 = k_1\boldsymbol{Q}_3 \end{cases} , \tag{4.76}$$

$$\therefore\ (k_0V_{123} - k_1V_{023})\ \boldsymbol{Q}_0 - (k_0V_{023} - k_1V_{013})\ \boldsymbol{Q}_1 + (k_0V_{013} - k_1V_{012})\ \boldsymbol{Q}_2 = \mathbf{0}. \tag{4.77}$$

Here, three points $\boldsymbol{Q}_0$, $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$ are not collinear; if so, the point $\boldsymbol{Q}_3$ is also on the same line by Eqn. (4.76). Thus,

$$\begin{cases} k_0V_{123} - k_1V_{023} = 0 \\ k_0V_{023} - k_1V_{013} = 0 \\ k_0V_{013} - k_1V_{012} = 0 \end{cases} , \tag{4.78}$$

and

$$k_0 \neq 0, \quad k_1 \neq 0,$$

$$\therefore \frac{V_{123}}{V_{023}} = \frac{V_{023}}{V_{013}} = \frac{V_{013}}{V_{012}} = \frac{k_1}{k_0}. \tag{4.79}$$

From Eqn. (4.7),

$$V_{012} = 9\boldsymbol{P}_0 \cdot (\boldsymbol{P}_1 \times \boldsymbol{P}_2)\,, \quad V_{013} = 3\boldsymbol{P}_0 \cdot (\boldsymbol{P}_1 \times \boldsymbol{P}_3)\,,$$
$$V_{023} = 3\boldsymbol{P}_0 \cdot (\boldsymbol{P}_2 \times \boldsymbol{P}_3)\,, \quad V_{123} = 9\boldsymbol{P}_1 \cdot (\boldsymbol{P}_2 \times \boldsymbol{P}_3) \tag{4.80}$$

Let S_{ijk}be the area of triangle $\Delta \boldsymbol{P}_i\boldsymbol{P}_j\boldsymbol{P}_k$, then,

$$S_{012} + S_{023} = S_{013} + S_{123} \tag{4.81}$$

Since the weight of each control point $\boldsymbol{P}_i$ is 1,

$$S_{ijk} = \frac{1}{2}\boldsymbol{P}_i \cdot (\boldsymbol{P}_j \times \boldsymbol{P}_k) \tag{4.82}$$

$$\therefore V_{012} + 3V_{023} = 3V_{013} + V_{123} \tag{4.83}$$

By solving Eqn. (5.1) and (4.83), we obtain the solutions:

$$\frac{k_1}{k_0} = 1, \quad -2 \pm \sqrt{3}. \tag{4.84}$$

However, k_1/k_0 must be positive; otherwise, two points $\boldsymbol{P}_2$ and $\boldsymbol{P}_3$ must be in the opposite sides of line

$\boldsymbol{P}_0\boldsymbol{P}_1$, two points $\boldsymbol{P}_0$ and $\boldsymbol{P}_3$ must be in the opposite sides of line $\boldsymbol{P}_1\boldsymbol{P}_2$, and two points $\boldsymbol{P}_0$ and $\boldsymbol{P}_1$ must be in the opposite sides of line $\boldsymbol{P}_2\boldsymbol{P}_3$, so it is impossible to locate the four control points. Therefore,

$$\frac{k_1}{k_0} = 1. \tag{4.85}$$

$$\therefore V_{012} = V_{013} = V_{023} = V_{123} \tag{4.86}$$

From Eqn. (4.76),

$$V_{012}\left(\boldsymbol{P}_0 - 3\boldsymbol{P}_1 + 3\boldsymbol{P}_2\right) = k_0\boldsymbol{P}_3. \tag{4.87}$$

Since the weight of each control point $\boldsymbol{P}_i$ is 1,

$$V_{012} = k_0 \tag{4.88}$$

$$\therefore 3\boldsymbol{P}_1 - \boldsymbol{P}_0 = 3\boldsymbol{P}_2 - \boldsymbol{P}_3. \tag{4.89}$$

Eqn. (4.89) shows that the curve $\boldsymbol{P}(t)$ is degree reducible; $\boldsymbol{P}(t)$ is equivalent to the quadratic Bézier curve with following control points $\boldsymbol{R}_i$:

$$\boldsymbol{R}_0 = \boldsymbol{P}_0, \quad \boldsymbol{R}_1 = \frac{3}{2}\boldsymbol{P}_1 - \frac{1}{2}\boldsymbol{P}_0 = \frac{3}{2}\boldsymbol{P}_2 - \frac{1}{2}\boldsymbol{P}_3,$$
$$\boldsymbol{R}_2 = \boldsymbol{P}_3. \tag{4.90}$$

As a result, for any cubic Bézier curve $\boldsymbol{P}(t)$, unless it is degenerate, the linear moving line $\boldsymbol{L}_0(t)$ and the double point $\boldsymbol{D}$ can be obtained correctly.

In Eqn. (4.14), $\boldsymbol{L}_1(t)$ can always be calculated, and Eqn. (4.3) and (4.13) shows that the result $\boldsymbol{L}_1(t)$ must satisfy

$$\boldsymbol{L}_1(t) \cdot \boldsymbol{P}(t) = 0 \tag{4.91}$$

If the latter process fails, $\boldsymbol{L}_1(t)$ must satisfy one of the following conditions:

(1) $\boldsymbol{L}_1(t) = \boldsymbol{0}$ (for $\forall t$),
(2) $\boldsymbol{L}_0(t)$ and $\boldsymbol{L}_1(t)$ are equivalent *i.e.* there exists a scalar function $k(t)$ that satisfies

$$\boldsymbol{L}_1(t) = k(t)\,\boldsymbol{L}_0(t), \tag{4.92}$$

(3) $\boldsymbol{L}_1(t)$ is actually a linear moving line.

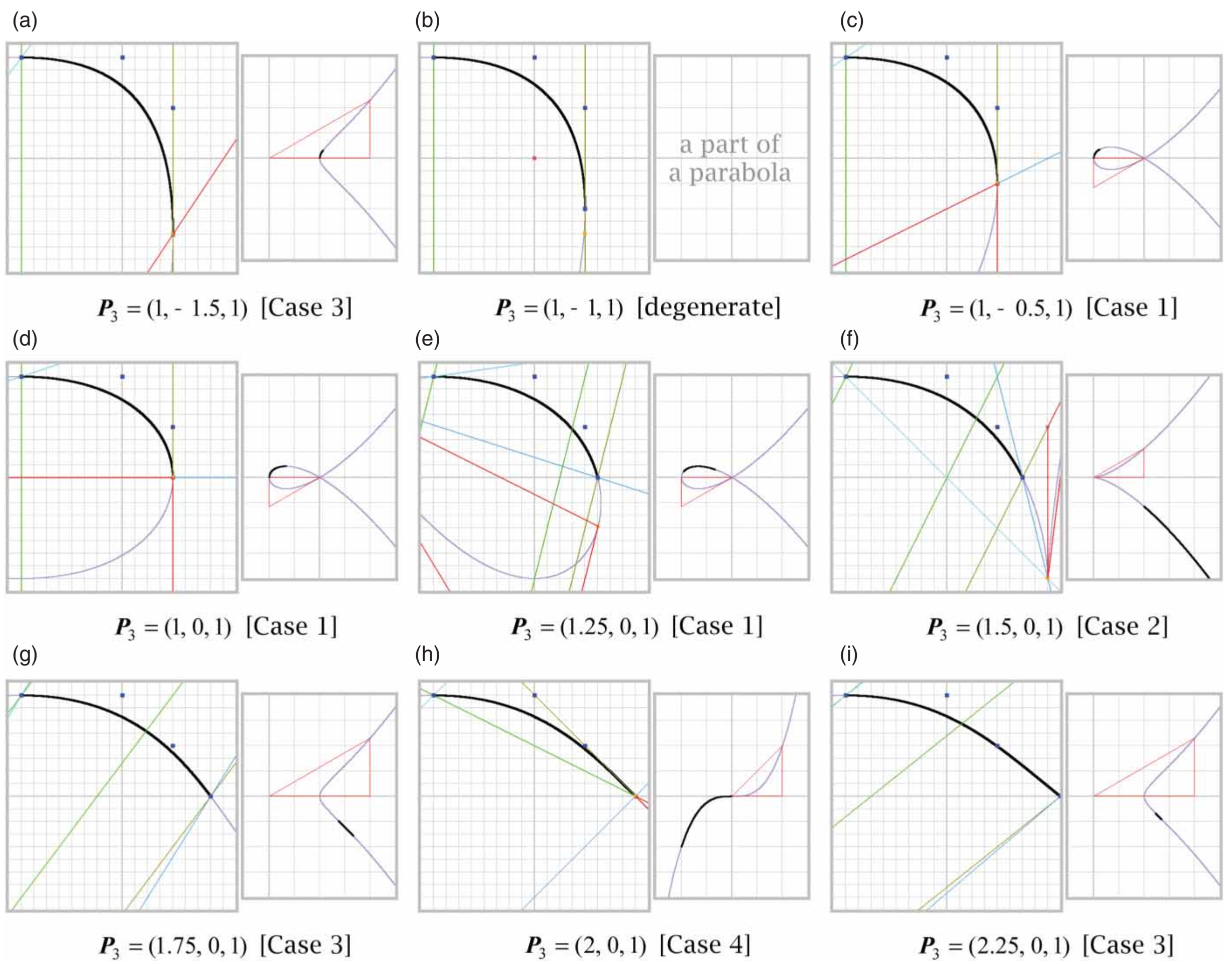

Fig. 8: Example Bézier curves and corresponding primitives.

(1) must not occur by the following reason. From Eqn. (4.14),

$$\boldsymbol{L}_{10} = -\boldsymbol{P}_0 \times (\boldsymbol{P}_3 - 3\,\boldsymbol{P}_2 + 3\,\boldsymbol{P}_1 - \boldsymbol{P}_0)\,, \tag{4.93}$$

Where $\boldsymbol{P}_0$ is a control point that is not at infinity and $(\boldsymbol{P}_3 - 3\,\boldsymbol{P}_2 + 3\,\boldsymbol{P}_1 - \boldsymbol{P}_0)$ is a point at infinity. Thus,

$$\boldsymbol{L}_1(0) = \boldsymbol{L}_{10} \neq \boldsymbol{0}. \tag{4.94}$$

(2) is impossible, because $\boldsymbol{L}_0(t)$ contains a fixed point $\boldsymbol{D}$ which is not at infinity, and $\boldsymbol{L}_1(t)$ contains a fixed point $\boldsymbol{P}(\infty)$ at infinity. If (3) is true, the curve $\boldsymbol{P}(t)$ must be degree reducible. Thus, (3) is also impossible. As a result, for any cubic Bézier curve $\boldsymbol{P}(t)$, unless it is degenerate or the double point $\boldsymbol{D}$ is at infinity, it is classified into one of the three cases in subsection 4.3.

5.2. Implementation and Experimental Results

We have implemented a program to check the theory in section 4. From given Bézier control points, the program calculates and displays control lines of the moving lines, reference points, corresponding primitive, transformation matrix, and parameter interval. Since each process is based on closed-form expressions, the computation is in real-time on recent PC's. Fig. 8 shows example results, where three control points are fixed and the other one is moved. For each Bezier curve, corresponding primitive and parameter interval (black part on the curve) are presented. Also, we have tried to display the Bézier curve from the result (primitive, transformation matrix, and parameter interval), and confirmed that it is always coincident with the input Bézier curve.

Three control points $\boldsymbol{P}_0 = (-2, 2, 1)$, $\boldsymbol{P}_1 = (0, 2, 1)$ and $\boldsymbol{P}_2 = (1, 1, 1)$ are fixed, and the other one $\boldsymbol{P}_3$ moves from (1, −1.5, 1) to (1, 0, 1), and the to (2.25, 0, 1).

5.3. Selection of Primitive Cubics

The selection of four primitive cubics in Fig. 1 is not unique. Each primitive can be replaced with any affine transformed one. In general, symmetrical shapes with simple coefficients in both implicit and parametric forms would be better for primitives. One of the simplest sets is:

$$x^3 + x^2 - y^2 = 0, \quad x^3 - y^2 = 0, \quad x^3 - x^2 - y^2 = 0, \quad \text{and} \quad x^3 - y = 0. \tag{4.95}$$

The reason we chose the set in Fig. 1 instead of Eqn. (4.95) is that the first primitive $x^3 + x^2 - 3y^2 = 0$ is a Pythagorean hodograph curve and its curvature can be presented in simpler form. It may help curvature analysis in the next step of our project.

6. CONCLUSION

In this paper, we presented that any planar polynomial cubic Bézier curve can be described as an affine transformation of a part of four primitive cubics, and propose an algorithm to derive the transformation matrix. By using the linear and the quadratic parallel moving lines that follow the Bézier curve, we can determine the type of the double point, from which the curve is classified into four cases: crunode, cusp, acnode, and explicit cubic. We confirmed that the proposed algorithm never fails unless the curve is degenerate.

Next, we will analyze the curvature properties of the four primitive cubics and their affine transformation. From this analysis, we will try to show the aesthetic ability of cubic curves. The extension to rational cubic Bézier is also the future study.

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