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Filling the Empty Spaces of the Sierpinski Tetrahedron to Create a 3D Puzzle

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ABSTRACT

The purpose of this paper is to present the method used to fill, with tetrahedrons of dimension $\frac{L}{4}$, the empty spaces of the Sierpinski tetrahedron. And the aim of this research is to create a 3D puzzle that is assembled using modules formed of tetrahedrons that bind and/or intersect each other. In this paper we illustrate the process that took place to generate the modules taking in to account the unions and intersections of the tetrahedrons and we use applications CAD-CAM-CAE to perform, from the modeling of the modules till the rapid prototyping, the 3D puzzle.

Keywords: Sierpinski tetrahedron, puzzle.

1. INTRODUCTION

In 1874 George Cantor published his first paper about set theory which shows that the set of integers has the same number of elements as the set of even numbers, and that the number of points in a segment is equal to an infinite line in space.

The Cantor set begins with a closed interval [0, 1] that it divides into three open subintervals of the same amplitude, the central third is removed and are preserved the two closed intervals of the first level of subdivision (Fig. 1(a)). Next, it divide in three open subintervals the two remaining closed intervals and from each the central third are remove and are preserved four closed intervals for the second level of subdivision (Fig. 1(b)).

It continues with the subdivide of each of the closed remaining intervals, is removed the central third to reach the third level (Fig. 2) and so on up to infinity.

Waclaw Sierpinski was known for outstanding contributions to set theory, number theory, theory of functions and topology and in 1919 introduced a fractal called Sierpinski triangle.

The Sierpinski triangle develops through recursive subdivision of equilateral triangles where, the interaction $n = 0$ is the equilateral triangle of side L, the interaction $n = 1$ is the midpoint of side L, i.e. $\frac{L}{2}$, the interaction $n = 2$ is where each of the triangles has a length $\frac{L}{4}$, and so on (Fig. 3).

The Sierpinski tetrahedron is the three-dimensional shape of the Sierpinski triangle where, in the interaction $n = 0$ there is one tetrahedron, in the interaction $n = 1$ there are four tetrahedrons, in the interaction $n = 3$ there are sixteen tetrahedrons an so on (Fig. 4).

This project stems from the idea of filling, with tetrahedral dimension $\frac{L}{4}(\frac{69.282}{4}mm)$, the empty spaces of Sierpinski tetrahedron in its second stage, i.e. in the $n = 2$ interaction, to build a 3D puzzle using the following methodology.

- Development of the Sierpinski tetrahedron in its second stage.
- Analysis of unions and intersections of the tetrahedrons.
- Generation of the modules.
- Assembly of the 3D puzzle.

The aim of this research is to create a 3D puzzle which is assembled using modules formed of tetrahedrons that bind and/or intersect each other. To achieve the objective of this research, in this study were considered: (a) the intersection of the geometric shapes, we understand as intersection, the meeting of two lines, two planes or two solids, which cutting each other, and (b) the construction of modules to get to assembly the 3D puzzle.

This paper is organized as follows: in Section 2 we give an explanation of the development of the Sierpinski tetrahedron in its second stage, in Section 3 we give an explanation of the analysis of unions and intersections of the tetrahedrons, in Section 4, we

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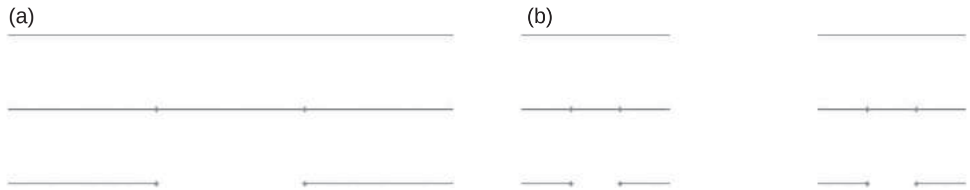

Fig. 1: Cantor set: (a) first level, (b) second level.

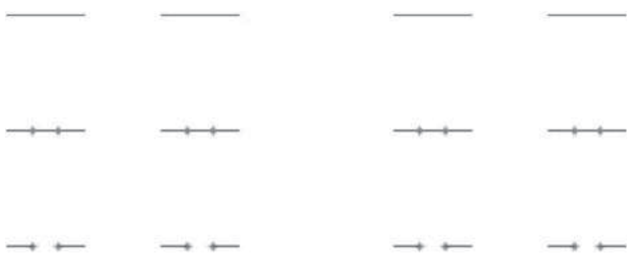

Fig. 2: Cantor set third level.

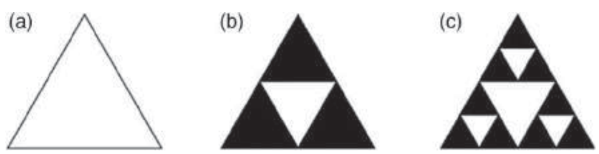

Fig. 3: Interactions of Sierpinski triangle: (a) n = 0, (b) n = 1 and (c) n = 2.

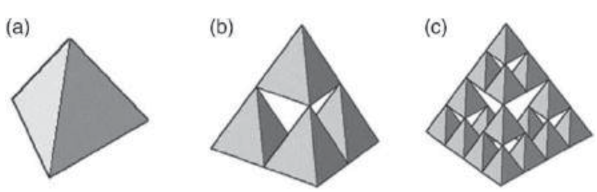

Fig. 4: Interaction of Sierpinski tetrahedron: (a) n = 0, (b) n = 1 and (c) n = 2.

explain the generation of the modules. In Section 5 we explain as the puzzle is assembled. Section 6 presents the results. Section 7 presents the use of this work in education and finally Section 8 the conclusions. We want to mention that all the figures presented in this paper are original and created by the authors at the Autonomous Metropolitan University, Cuajimalpa in Mexico City.

2. DEVELOPMENT OF SIERPINSKI TETRAHEDRON IN ITS SECOND STAGE

The tetrahedron is a polyhedron formed by four equilateral triangles and four vertices. In Book XI of Euclid's Elements is considered the tetrahedron as "A solid figure bounded by planes that are formed from a plane at an arbitrary point." The tetrahedron is one of the five perfect polyhedra called Platonic solids, is one of the eight convex polyhedra called deltahedra (polyhedron whose faces are equal equilateral triangles), and complies with the theorem of Euler polyhedra $4 + 4 = 6 + 2$ ($c + v = a + 2$).

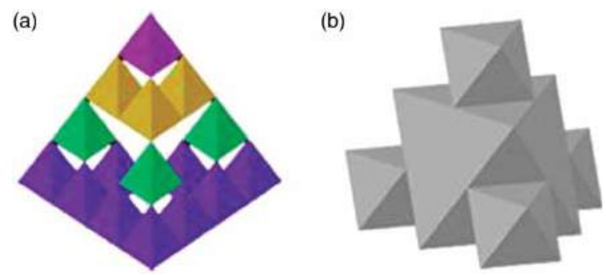

Fig. 5: Sierpinski tetrahedron: (a) second stage (b) irregular shapes of the empty spaces.

In developing the Sierpinski tetrahedron in its second stage are used sixteen tetrahedrons dimension $\frac{L}{4}$ which bind in to four modules with four tetrahedrons each (Fig. 5(a)), so if the distance of 69,282 mm of the edge of the tetrahedron is divided into four parts, the edge of each of the tetrahedrons will be 17.3205 mm.

The Sierpinski tetrahedron has 75% of empty spaces of equal irregular shapes of different sizes (Fig. 5(b)), because the sixteen tetrahedrons are removed from the initial tetrahedron (interaction n = 0).

The tetrahedron of edge 69.282 mm has a volume of 39.192 cm^3 and each of the sixteen tetrahedrons has a volume of 0.612 cm^3, i.e. a total of 9.797 cm^3, so we have a volume of 29.395 cm^3 of empty space, i.e. the 75% of empty space.

3. ANALYSIS OF UNIONS AND INTERSECTIONS OF THE TETRAHEDRONS

3.1. Unions of the Tetrahedrons

The etymological origin of the term union comes from the word *unus*, which can be translated as "one". Union is the action and effect of joining (putting together, combine) and to join two or more tetrahedrons it must consider the position in space of the tetrahedron and the inclination of its faces.

There can only be two unions between the tetrahedrons: the first between the faces that are parallel to the horizontal plane of the orthogonal projection (Fig. 6(a)) and the second between the inclined faces (Fig. 6(b)).

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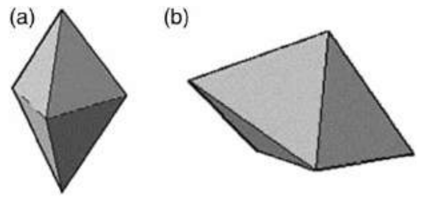

Fig. 6: Unions: (a) between horizontal faces, (b) between inclined faces.

3.2. Intersections of the Tetrahedrons

The intersection can be defined as the region of space which is occupied simultaneously by two or more shapes. The intersection can be used to tie two or more shapes or to create an empty space where a geometric shape can be inserted into the open space.

3.2.1. Intersections that tie two or more tetrahedrons

Depending on the position of the tetrahedrons in space is the manner in which they intersect. At the figures below (Fig. 7 and 8) it can see the five ways to make the intersections that tie two or three tetrahedrons respectively.

3.2.2. Intersections that create an empty space

There are several ways to make the intersections between two or more tetrahedrons so that they can create empty spaces, but in this study only took into account the below intersections because they are the ones that will be used to assemble the puzzle. At the figures below (Fig. 9) it can see the four ways to make the intersections where an empty space is created so that a tetrahedron can be inserted into the open space.

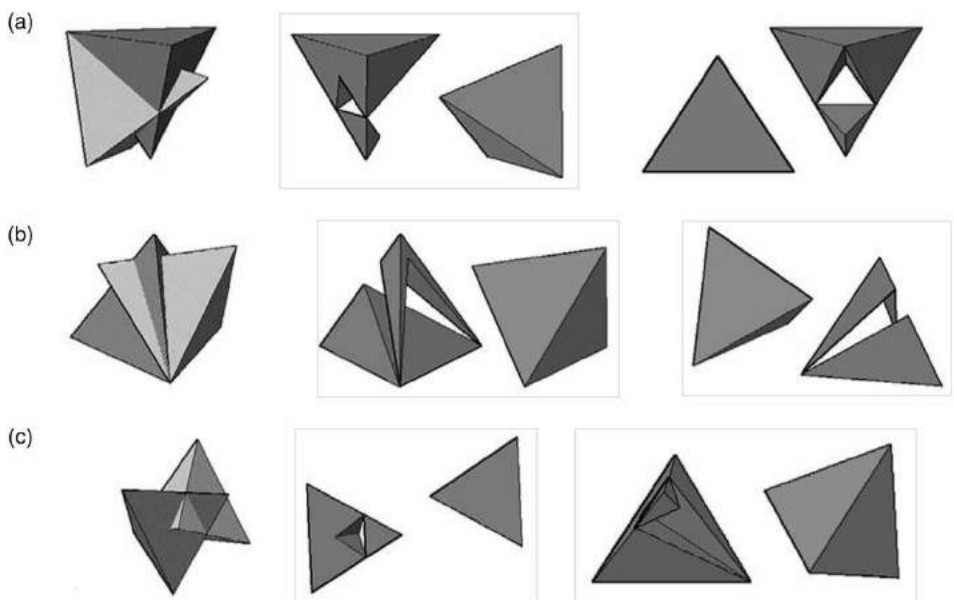

Fig. 7: Intersection of two tetrahedrons: (a) first way, (b) second way, (c) third way.

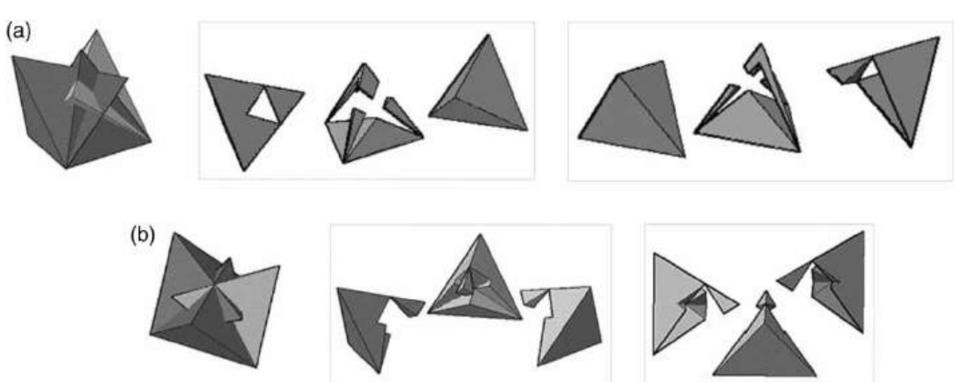

Fig. 8: Intersection of three tetrahedrons: (a) first way, (b) second way.

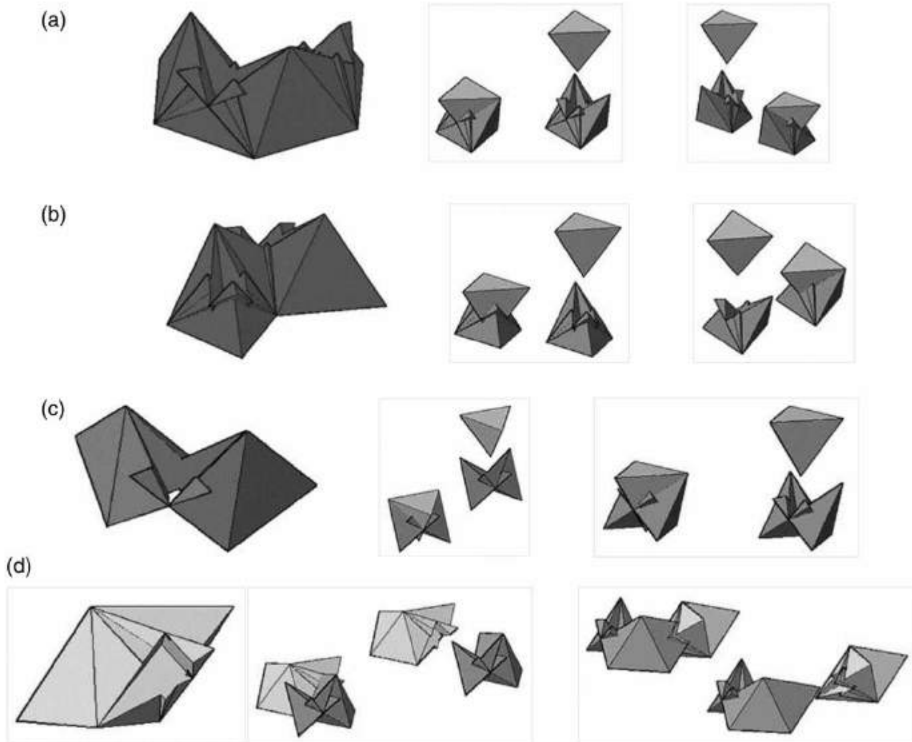

Fig. 9: Intersection through empty spaces: (a) first, (b) second, (c) third and (d) fourth way.

4. GENERATION OF THE MODULES

A module is defined as each of the separate parts that make a whole. Our design is formed by ten modules with different numbers of tetrahedrons that bind and/or intersect each other (Tab. 1) and are arranged in three levels (Fig. 10(a)).

Number of modules	Number of tetrahedrons
2	2
4	4
1	5
2	7
1	8
Total	Total
10	47

Tab. 1: Modules.

Before creating the modules, it was considered the arrangement of seven tetrahedrons to form, at the base of the first level, an equilateral triangle, where a tetrahedron is placed in the center of the equilateral triangle and six tetrahedrons were placed around it so that the tetrahedrons are connected by the vertices (Fig. 10(b)).

To better understand how the modules were generated, each tetrahedra will have a number for identification that can be relate them in the tables (Tab. 2,

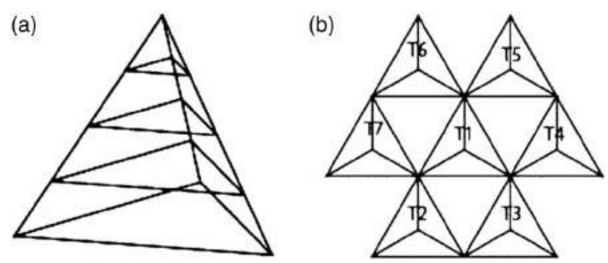

Fig. 10: (a) Levels and (b) seven tetrahedrons at first level.

3 and 4) and in the figures (Fig. 11, 12, 13, 14 and 15) which are presented below.

4.1. Modules first Level

The first level consists of four modules which are generated with different numbers of tetrahedrons that bind and/or intersect each other. Table 2 explains, which tetrahedrons (T) and modules intersecting (I) with other tetrahedrons and with other modules of the same level or second level.

4.2. Modules Second Level

The second level consists of five modules which are generated with different numbers of tetrahedrons that bind and/or intersect each other. Table 3

Modules	Tetrahedrons	Faces which bind	Tetrahedron that intersect.	Intersecting modules	Edges that join
1	T1, T8, T9,T10	T1-T8 and T1-T9	T9-T10 (I1)	Module 2 I1-T13 Module 8 I9-T8	T1-T18
2	T3, T11, T12, T4 and T13	T3-T11; T12-T4 and T4-T13	T11-T12 (I2)	Module 1 T13- I1 Module 7 I2-T27	T3-T17
3	T7,T14,T15,T16,T17,T18 and T2	T7-T14; T2-T16 and T2-T17	T14-T15-T16 (I3) T17-T18 (I4)	Module 5 I3-T21 Module 6 I4-T23	T18-T1 T17-T3
4	T6,T19,T20 and T5	T6-T20 and T5-T19	T20-T19 (I5)	Module 8 I5-T31	NONE

Tab. 2: Modules first level related with Figures 11 and 12.

Modules	Tetrahedrons	Faces which bind	Tetrahedron that intersect.	Intersecting modules	Edges that join
5	T21 and T22	T21-T22	NONE	Module 3 I3-T21	T21-T22-T26
6	T23, T24, T25 and T26	T23-T24 and T24-T25	T25-T26 (I6)	Module 3 I4-T23 Module 10 I6-T41	T26-T21-T22
7	T27,T28,729 and T30	T27-T28 and T28-T29	T29-T30 (I7)	Module 2 T27-I2	T44-T30
8	T31,T32,T33, T34,T35,T36, T37 and T38	T31-T32, T32-T33, T32-T34, T34-T36, T36-T37	T33-T34-T35 (I8) T37-T38 (I9)	Module 4 T31-I5 Module 1 I9-T8 Module 9 I8-T39	T38-T20 T43-T34
9	T39 and T40	T39-T40	NONE	Module 8 I8-T39	NONE

Tab. 3: Modules second level related with Figures 13 and 14.

Modules	Tetrahedrons	Faces which bind	Tetrahedron that intersect.	Intersecting modules	Edges that join
10	T41, T42, T43, T44, T45, T46 and T47	T41-T42, T42-T46, T43-T47 and T44-T45	T45-T46-T47 (I10)	Module 6 I6-T41	T43-T34 T44-T30

Tab. 4: Module third level related with Figure 15.

explains, which tetrahedrons (T) and modules intersecting (I) with other tetrahedrons and with other modules of the same level or first level.

4.3. Module Third Level

The third level consists of one module which is generated with seven tetrahedrons that bind and/or intersect each other. In Table 4 is explained which tetrahedrons (T) and module intersecting (I) with other tetrahedrons and with the module from the second level.

5. ASSEMBLY OF THE 3D PUZZLE

The assembly word is derived from the French verb "assembler" and is defined as the union of two pieces forming part of a structure. There are many ways to

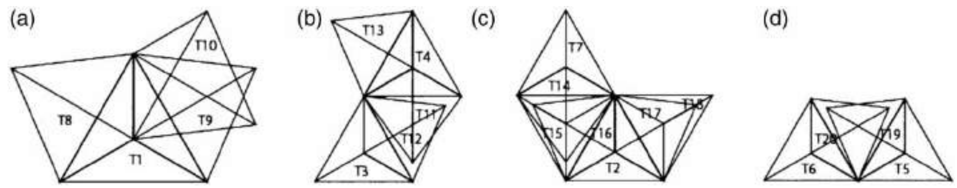

Fig. 11: Modules first level: (a) first, (b) second, (c) third and (d) fourth.

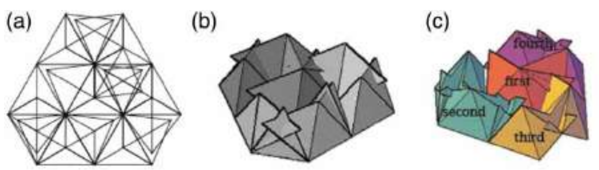

Fig. 12: Modules first level: (a) orthogonal projection, (b) three-dimensional graphical representation and (c) unions, intersections and empty spaces.

assemble this puzzle but it is suggested to follow an order in the placing of the modules since otherwise would be very difficult to introduce the tetrahedrons at the empty spaces. The following figures (Fig. 16 and 17) present the prototype that was printed in the university laboratory and it can see how it was assembled the puzzle.

Table 5 indicates the modules that should be inserted in the empty spaces as a reference to assemble the 3D puzzle.

6. RESULTS

Some of the 2D puzzles made in the course of history are the Stomachion, described in the manuscripts made by Archimedes, the Tangram that is an ancient Chinese game called the Chi Chiao Pan, the Pentominos, and the wood puzzles carried out in 1760 by John Spilsbury.

The 3D puzzles began with the shape of a cube, for example, the Soma Cube, the Lesk Cube, the O'Beirne's Cube and the most famous Rubik's Cube.

The first 3D mechanical tetrahedral puzzles were designed by Rubik (Pyramorphix) and Meffert (Pyraminx) followed by Adam Zamora (Megamorphix), by Greenhill, and by Frederic Plateús, to name a few, and also we find the 3D tetrahedral puzzles designed by James Allwright [7] and Wayne Daniel [8], who used modules to assemble its 3D puzzles.

Fig. 14: Modules second level: (a) orthogonal projection, (b) three-dimensional graphical representation and (c) unions, intersections and empty spaces.

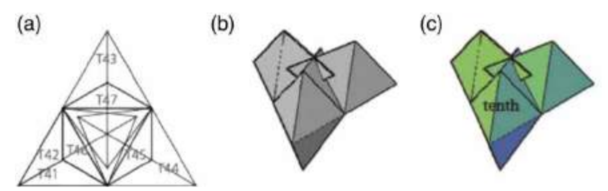

Fig. 15: Module third level: (a) tenth, (b) three-dimensional graphical representation and (c) unions and intersections.

Fig. 16: The ten modules printed.

These latest puzzles, which are those that will relate to our research, are made through cuts and intersections of the tetrahedron, for example, Allwright intersects truncated tetrahedra to assemble the puzzle, and Daniel cut the tetrahedron into two, three or four equal pieces or in nine different pieces to assemble their puzzles.

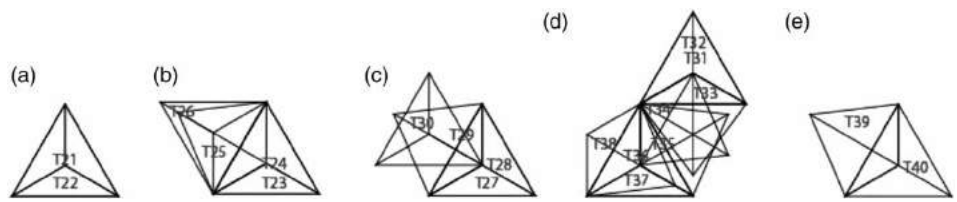

Fig. 13: Modules second level: (a) fifth, (b) sixth, (c) seventh, (d) eighth and (e) ninth.

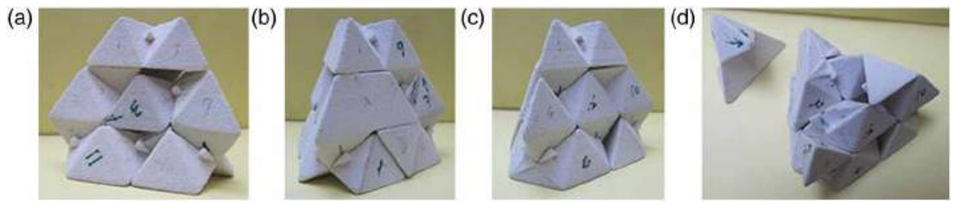

Fig. 17: 3D puzzle, (a, b, c) exterior views and (d) interior view.

Modules with empty spaces	Modules that insert in the empty spaces
6	10
3	7
2	5 and 6
1	3
4	8
8	1 and 9

Tab. 5: Modules insertion.

The difference between the 3D tetrahedral puzzles designed by Allwright and Daniel and the 3D puzzle presented in this research is that the tetrahedrons in each of the modules is based on the dimension $\frac{L}{4}$ of the Sierpinski tetrahedron, i.e. one quarter of the length of the edge, and in each of the modules took into account the unions between two tetrahedrons, the intersections between two or three tetrahedrons, and the empty spaces to insert a tetrahedron for assembling the 3D puzzle.

Once assembled the 3d puzzle, as indicates in the Table 5, we find that certain edges, faces and vertices of the tetrahedrons are joined (Tab. 6), the 75% of empty spaces of Sierpinski tetrahedron is reduced to 32.2% in the 3D puzzle, i.e., of 29.395 cm^3 to 12.615 cm^3 (Fig. 18, 19 and 20), and we have 14.15%, i.e. 5.549 cm^3, of interior empty spaces in the 3D puzzle (Fig. 21).

Regardless that have not filled in all the empty spaces of Sierpinski tetrahedron and due to the inclination of the faces of the tetrahedrons, the goal of this research is achieved, create a 3D puzzle that is assembled using modules formed of tetrahedrons that bind and/or intersect each other (Fig. 22 and Fig. 23).

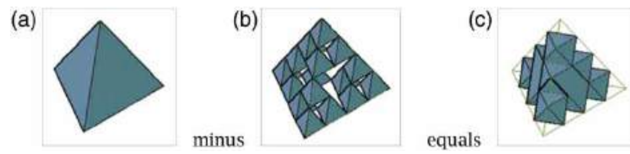

Fig. 18: Empty spaces: (a) tetrahedron, (b) sierpinski tetrahedron, (c) 75% of empty spaces.

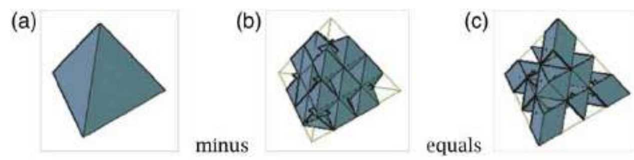

Fig. 19: Empty spaces: (a) tetrahedron, (b) 3D puzzle (c) 38.43% of empty spaces.

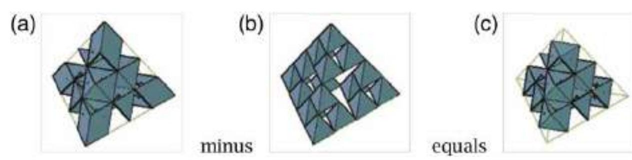

Fig. 20: Empty spaces: (a) 38.43% of empty spaces, (b) Sierpinski tetrahedron, (c) 32.2% of empty spaces.

7. USE OF THIS WORK IN EDUCATION

A puzzle is not just a game. It is an important educational learning tool that provides the development of many mental skills, such as: capacity for analysis and synthesis, coordination, spatial vision, motor skills, logical thinking and creativity.

		Joined at the faces		Joined at the vertices	
Modules	Modules	Modules	Modules	Modules	Modules
1	**3**	**10**	**7**	**4**	**3, 1, 2 6 and 9**
8	6 and 3			6	4, 2, 7 and 9
				1	8
				9	4, 2 and 6
				10	5, 8 and 7

Tab. 6: Modules joined: (a) at the edges, (b) at the faces and (c) at the vertices.

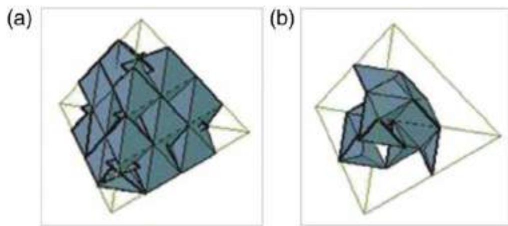

Fig. 21: Interior empty spaces: (a) 3D puzzle, (b) 14.51% of empty spaces.

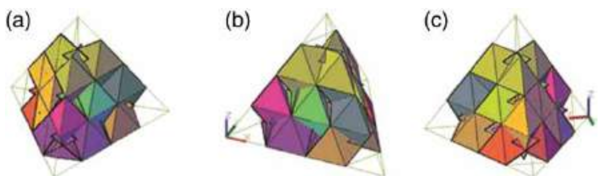

Fig. 22: 3d puzzle: (a, b, c) views.

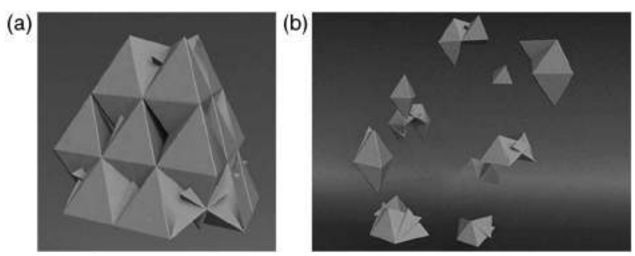

Fig. 23: 3D puzzle: (a) render and (b) explosion.

To assemble any puzzle you need to have the instructions, so were given the students studying Masters in “Design, Information and Communication” at the Universidad Autonoma Metropolitana, Cuajimalpa, the 3D puzzle of this work so that they carry out the instruction manual.

As we can see in the Figures 24 and 25(a. b and c), the students analyze each of the printed modules and

Fig. 26: First page of the printed instruction manual.

the Autocad file, they assemble the puzzle, and they write the instructions.

Students concluded that if they paint each of the modules of different color (Fig. 25(d) it will be easier to assemble the puzzle and that this puzzle needs two manuals, one printed (Fig. 26) and the other through a movie with sound. After evaluating the manuals, the results showed that it is possible to assemble the puzzle by following the instructions, concluding that the instructions are well illustrated and explained.

8. CONCLUSION

This paper presents a system to create a 3D puzzle by filling the empty spaces of the Sierpinski tetrahedron. The puzzle is made up of a number of modules and the modules are a series of tetrahedrons of dimension $\frac{L}{4}$ forming different shapes.

The results show that it is not possible to create a 3D puzzle completely filled with tetrahedrons of dimension $\frac{L}{4}$, because the tetrahedron is a solid formed by four equilateral triangles that meet at a vertex, and always, no matter the position of the tetrahedron in space maintains its inclination angle,

Fig. 24: Students working in the laboratory of the university.

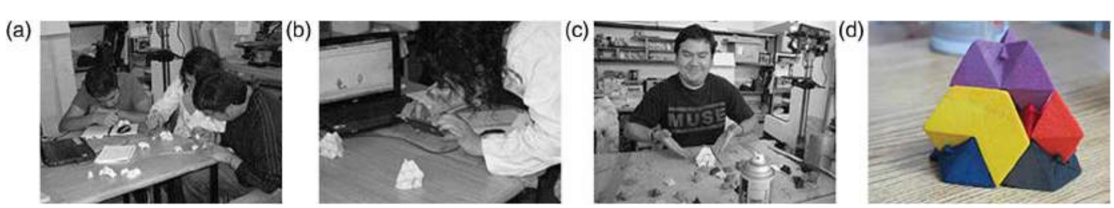

Fig. 25: (a, b and c) Students working in the laboratory of the university, (d) modules of different color.

and due to this inclination was achieved that the 3D puzzle can be assembled.

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