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Geometric Optimization for Developable Panel Approximation

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ABSTRACT

This paper presents a type of surface approximation problem that commonly arises in relation to design for manufacturing of complex architectural surfaces. The engineering context of the problem is introduced by a real-world example. The problem requires approximation of each facet of a surface subdivision by a simpler surface that is economical to fabricate into a panel. However, apart from the fitting accuracy, it is also required to manage the gaps between neighboring panels. We show that the concept of shape features is useful in modeling such problems into a global optimization problem. When such non-linear optimization formulations can be solved, we obtain better solutions that the two stage method currently used by the industry.

Keywords: developable surface approximation, shape features.

1. BACKGROUND

Recent years have seen a profusion of complex architectural projects across the world. Examples include the Al Raha Race Track in Abu Dhabi and the Dongdaemun Plaza (DDP) in Seoul (Fig. 1). Firms of leading architects such as Frank Gehry and Zaha Hadid are increasingly employing free-from surfaces in their designs.

The design of the building skin in such projects goes through several stages. Initially the architects produce a geometric model using surface design tools. The 3D CAD surfaces may be produced using any shape operator (e.g. subdivision-based approaches and T-Splines, skinning or lofting, curve sweeps, etc.), but the end result is typically a piecewise B-Spline surface. Next, the architect defines the panelization by generating a pattern of subdividing curves on the surface. These two steps essentially dictate the aesthetics of the building. The next stage is to determine the engineering details, such as the geometric constraints on each panel (e.g. four-sided planar, triangular, non-planar, etc.), the material of the panels (e.g., glass, titanium, aluminum, stainless steel, etc.), and the configuration of the substructure that will be used to support the panels (such structural elements are made of a mesh of steel bars). This is followed by a stage of geometric refinement – in this stage, the surface geometry as well as the subdividing curves are allowed to be perturbed, such that the individual panels are cheaper for fabricate while preserving the aesthetics. This stage is referred to as *rationalization*. Finally, the engineering design is performed, in which the exact dimensions of individual members are determined and their locations in the global coordinate frame fixed. Each stage uses the result of the previous one as its input.

In this paper, we propose that significant advantages can be derived by combining the last two steps of the design process into single optimization model. This approach shall be illustrated by means of a case study from the Kai Tak Cruise Terminal building in Hong Kong. Most of the building is covered with aluminum panels defined by developable surfaces. However, in some regions, e.g. along the curved lip of the terminal roof (see Fig. 2), the shape is more complex. The approach adopted in the original design was to subdivide the entire lip into panels of approximately equal length, and then approximating each such piece by developable shapes. We applied our proposed approach in an attempt to improve the panel rationalization.

We begin with a brief recap of related research in section 2. Section 3 presents details of our approach; section 4 is a case study describing our implementation and its application to a real world problem. The final section closes the paper with a brief conclusion and discussion.

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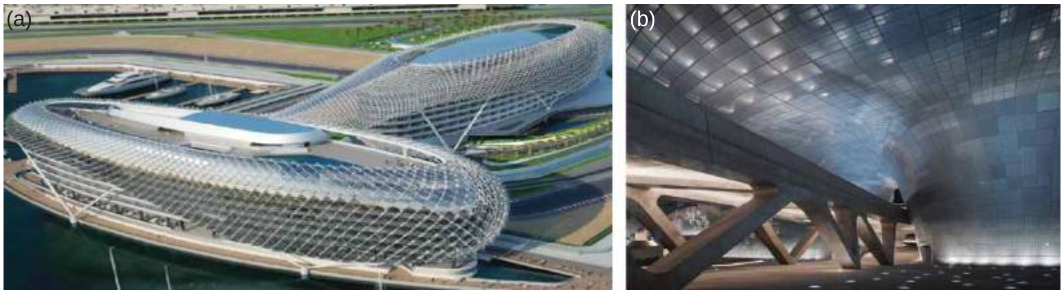

Fig. 1: (a) Al Raha race track, Abu Dhabi, (b) The interior panelization in the DDP, Seoul.

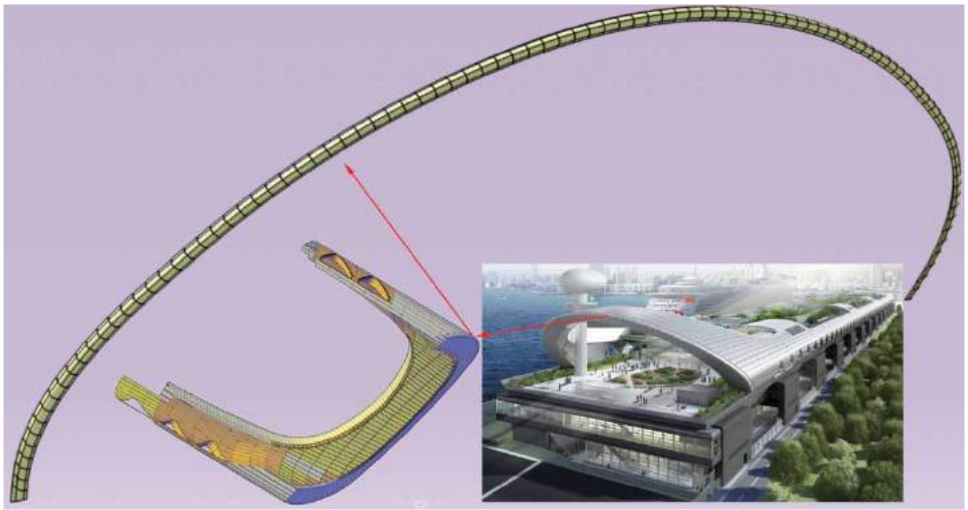

Fig. 2: The Kai Tak Cruise Terminal Building (architect: Norman Foster). The CAD model of the front region of the building is shown in the inset, and the expanded view of the panels making up the arch shaped lip of the terminal roof.

2. RELATED WORK

Approximating a free-form surface by piecewise developable patches is a classical problem faced in many areas of engineering such as ship building, textiles and footwear manufacture, etc. Two popular approaches for these problems are: (a) generate the initial design independent of any manufacturability constraints, and later modify the shape in some minimal way into an approximation that is piecewise developable, while satisfying other constraints (see for example [6] and [9]); or (b) restrict the designer by only allowing the use of developable shapes during the design (an approach recently discussed in [8]).

A well-studied strategy using the first approach is the problem of approximating a given surface by a ruled surface. A ruled surface is one in which the tangent line at any point is entirely contained within the surface. In practice, a common method for defining a ruled surface is via a linear interpolation between two space curves. If the curves are defined by B-splines, then the ruled surface can be expressed as:

$$S(u,\ v) = \sum_{i=0}^{n} N_{i,k}(u)(1-v)P_{i,0} + \sum_{i=0}^{n} N_{i,k}(u)\,vP_{i,1} \quad (1)$$

where $P_{i,0}$, (respectively, $P_{i,1}$) are the homogeneous coordinates (given by $[w_{ij}\,x_{ij},\ w_{ij}\,y_{ij},\ w_{ij}\,z_{ij},\ w_{ij}]$ for some non-zero weight w_{ij}) of the control points of the boundary curve of the surface, and $N_{i,k}$ are the Bernstein basis functions of order *k*. Hoschek *et al.* [2] used point-plane duality in projective space to represent a ruled surface as the envelop of a family of planes. In particular, the tangent plane at each point along the generator curve (which contains the ruling at that point) is represented as a point in projective space. This family of planes can be expressed as:

$$Y(u,\ v) = \sum_{i=0}^{n} N_{i,k}(u)(1-v)U_{i,0} + \sum_{i=0}^{n} N_{i,k}(u)\,vU_{i,1} \quad (2)$$

where $U_{i,j}$ are the homogeneous plane coordinates of the tangent planes at the knot point *i,j*. The boundary ruling at $u = 0$ is thus the intersection of the two tangent planes $U_{0,0}$ and $U_{0,1}$. In particular, if these two planes are coincident, then the ruled surface is developable in the neighborhood of this ruling. If $U_{u,0}$ and $U_{u,1}$ coincide for all *u*, then the surface is developable. The approach presented by Hoschek *et al.*

expresses the tangent disparity at some isoparametric points in terms of the L_2 norm of the distance between the two corresponding points in dual space. In order to approximate a given surface by a developable ruled surface, they sample a large number of points along the two bounding curves, and solve for an ordered indexing along these points that minimizes the sum of this error over entire surface. The idea of approximation of a ruled surface by a developable ruling was explored further by several researchers. In [5] an iterative algorithm is presented to solve this problem. The input is a pair of BSpline curves. The algorithm iteratively finds pairings of points, one from each curve, to form potential rulings. The objective is to minimize the warp angle (angular error between the tangent planes at the end points of a ruling. The matching that yields a good sequence of pairings is used to generate a re-parameterization of the input curves and consequently, the interpolating quasi-developable ruled surface. No guarantees are given about the convergence of the algorithm, or the interpolation tolerances. In [9], a variation of the approach of Hoschek was developed. Given a surface S(u, v) and a user defined curve c(u(t), v(t)) lying on the surface, a developable ruled surface approximation was generated that would minimize the Hausdorff distance between the original surface and the approximation. The Hausdorff distance between two point sets A and B is defined as:

$$D_H = \mathrm{dist}(A, B) = \max\{\max_{p\in A}\{\min_{q\in B}\{\|p - q\|\},\\ \max_{q\in B}\{\min_{p\in A}\{\|p - q\|\}\}$$

Since D_H is non-linear, [9] define a bound for the error, which is used in the subsequent approximation algorithm. The ruled surface is the envelop surface defined along c(t) as: E(t, r) = c(t) + r(N(t) X N'(t))/|| N(t) X N'(t) ||, where the prime represents the derivative with respect to the parameter *t*, and therefore the second term is merely the vector along the line on the tangent plane of S(u, v), orthogonal to the tangent of c(t). Since this equation is not a polynomial, it is approximated by a low order polynomial spline. The algorithm then searches for the best approximation ruling length by iteratively finding the farthest extension of the ruling that keeps error below a user-defined tolerance. In the end, a smoothing algorithm is used to refine the second bounding curve of the developable ruled surface. By repeatedly using the newly discovered bounding curve as an input, the entire surface can be approximated.

A more comprehensive work on developable approximation of BSpline surfaces was done by Pottman and Wallner [6]. They follow Hoschek's approach of describing a ruled surface as an interpolation of the tangent planes at the knot points. They further restrict the tangent planes to take the form U(t) = (u_0(t), u_1(t), t, −1), such that the derivative U'(t) = (u'_0(t), u'_1(t), 1, 0). This representation does not impose many restrictions on the form of the ruled surface, since the homogeneous coordinate = −1 only represents a scaling in E^3, and the restriction of the third coordinate is equivalent to a re-parameterization of the bounding curve. They represent the developable ruled surface as: S(t) = (1 − v) C_0(t) + vC_1(t), where C_0(t) and C_1(t) are planar intersection curves of the input BSpline surface, and therefore themselves BSpline curves. With some other restrictions on the variation of the orientation of the surface, they define a distance metric between two planes U_1, U_2 as the Lebesgue measure:

$$d_\mu(U_1 - U_2)^2 = \int_D ((u_{0,1} - u_{0,2}) + (u_{1,1} - u_{1,2})\,x \\ + (u_{2,1} - u_{2,2})\,y)^2\,dxdy$$

or, in a discretized sense with the distance being lumped as a mass at j discrete points, as:

$$d_\mu(U_1 - U_2)^2 = \sum_j ((u_{0,1} - u_{0,2}) + (u_{1,1} - u_{1,2})\,x_j \\ + (u_{2,1} - u_{2,2})\,y_j)^2$$

Suppose that the input surface is approximated by a series of *m* planes, $V_1 \ldots V_m$, then the required (unknown) developable ruled surface, U(t), can be discovered by solving the system:

Minimize $F = \sum_{i=1}^{m} d_{\mu i}(V_i, U(v_i))^2$. Furthermore, if the developable surface is of the form in Eqn. (2) and the bounding curves are of degree 3, then F is a quadratic function of the parameters of U_i, which can be therefore determined by solving a system of linear equations. This work was later extended to allow replacing the approximating planes V_i by a series of approximating conical strips in [1, 3].

All of the above work focused on the approximation of the input surface by a developable ruled surface. This is very useful in some industries, such as ship-building, where relatively few strips of steel need to be custom-fabricated for each project. However, in façade design, a typical building surface may require fabrication of a few thousand such curved panels; this fabrication is achieved via rolling machines like the one shown in Fig. 3 below. Such machines can easily fabricate cylindrical or conical panels, but are very cumbersome to control in order to produce arbitrary ruled surface strips.

Therefore it is more interesting for us to study surface approximation using general conical surfaces. That problem has also received some attention in some engineering contexts. A good introduction to robust fitting of a set of sampled points by cones (and other analytical surfaces) can be found in [4]. Since any of these surfaces can be defined in terms of a small number of parameters, when the set of sampled points is large, the problem is over-constrained. At the same time, there is inevitably some noise in the sampling data, and therefore almost all instances of such interpolation problems are solved by the use of some form of regression fit, typically requiring the solution to a set of low order non-linear polynomial equations. These systems are highly non-linear, and therefore when the data is not well behaved, numerical methods often fail to converge to a globally optimum solution. For example, if the input set of points is nearly planar and we wish to find the best fitting cylinder, the system may have local minima for vastly different axis directions. The main focus of works such as [4] is in developing robust algorithms that address such issues. [7] used a least squares regression model to fit cylinders into a set of points sampled from a free form surface. There have also been several papers in which the problem of approximating a developable ruled surface by a set of cones is studied [10, 3].

Fig. 3: A rolling machine used to produce cylindrical/conical panels.

However, in all of the previous work, each approximation, or fitting algorithm runs independently over its domain, with no additional constraints on the fitting surface. Furthermore, in some practical settings, such as the one that motivates our study, the fitting surface is a piecewise developable surface composed of up to three patches with G^1 continuity at their shared boundaries. Such panels are very common in modern architectural projects, and therefore a model to produce the best fit on such shapes is interesting. In the next section, we shall introduce the particular form of our problem, and present a simple approach for simultaneously fitting multiple surfaces with constraints along their boundaries.

3. PROBLEM DESCRIPTION AND METHODOLOGY

We are given a free form surface, together with a curve mesh on the surface inducing a subdivision. We are given a *shape feature* (defined below), describing a generic instance of the approximating shape, specified in terms of a small number of geometric parameters. Our objective is to approximate each face of the subdivision by an instance of the shape feature, such that the feature instances meet some user-defined constraints along their shared boundaries in addition to minimizing the approximation error.

In our context, we define a shape feature as a connected surface made up of one or more patches, each of which can de described by a single analytical equation. Each patch is bounded by a curve polygon with the same number of boundary edges. Each bounding edge has a well-defined form (e.g. planar curve, straight line, arc, etc.). Each patch is connected to one or more of its neighboring patches along boundary curves, with specified continuity condition. Some typical atomic shape features are shown in Fig. 4, and an example of a composite shape feature is shown in Fig. 5.

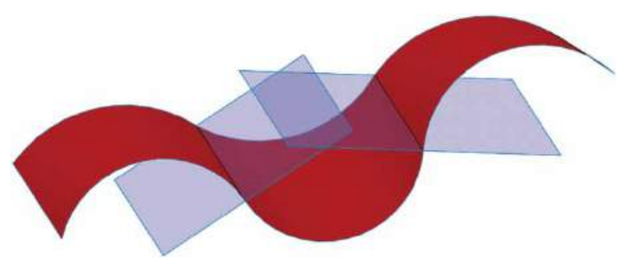

Fig. 5: A tri-cone shape feature (red color). The feature contains three patches, each of which is a right cone. Each patch is bounded by four lines, two of which are generatrices of the cone, and the other two are conic sections (and therefore planar). Neighboring patches are G^1 continuous along their shared boundary.

Fig. 6 illustrates the key issue in approximating a given surface patch by a feature instance. In this case, it can be seen that approximating the same surface by a plane or a cone results in very different approximation error.

Fitting an atomic feature to a given surface is typically achieved by sampling a sufficient number of

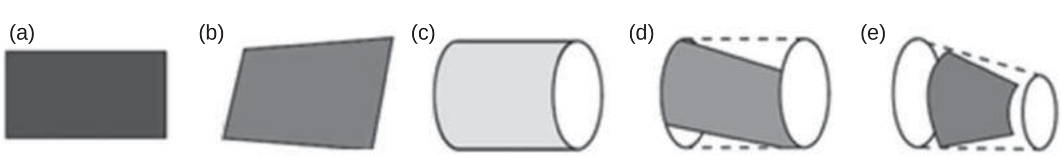

Fig. 4: Atomic shape feature instances: (a) planar rectangular (b) planar polygon (c) cylindrical parallel (d) cylindrical inclined (e) conical.

points spread over the entire surface (and possibly using a non-uniform density depending on the user requirements), and then solving a regression model. For example, a cone may be specified via six parameters (3 coordinates for the cone point, two angles to define the cone axis, and the cone half angle). Fig. 7 shows the expression for error of a sampled point from the surface of the cone. A regression model for this function could, for example, minimize the sum of the squared error, e_i^2, over all sampled points P_i. Solving the regression model requires solving a set of six non-linear equations simultaneously.

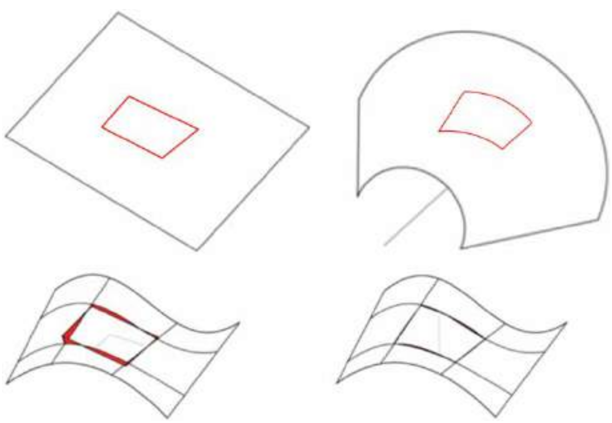

Fig. 6: Fitting a single surface subdivision with a planar or a conical patch. The fitting error is the Hausdorff distance between the approximation and the original surface for some optimum placement of the approximation in the coordinate frame of the original surface. The figure shows the error along the boundaries.

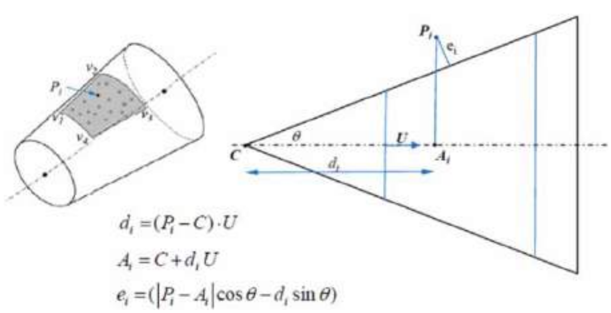

Fig. 7: Regression model setup for cone fitting.

In this research, the objective was mainly to test the efficacy of combining the two stages of this rationalization process: the fitting of feature instances to the faces of the subdivision and locating them relative to each other by solving a single optimization model simultaneously. Intuitively, it is clear that given a perfect solver, our approach searches over a superset of the solution space explored by the two-stage approach. In practice, there is a trade-off between the domain of the search and the ability of the numerical solver to hunt down the global optimum.

The case study for testing our approach comes from a sequence of aluminum panels form the lip region of the Kai Tak Cruise terminal project as indicated in Fig. 2. Fig. 8(a) shows the suggested subdivision of the surface into ideal panels specified by the architect. In this scheme, each panel can be approximated by a shape feature composed of three patches: two planar patches joined together by a curved patch in the middle. We set the template shape feature as a plane-cone-plane structure (see Fig. 9(b)), with G^1 continuity along the shared boundaries. It immediately follows that the boundary of the conical patch is defined by directrices of the cone (otherwise the boundary would not be a planar curve, and therefore the shared boundary with the plane would have discontinuity). The parametric representation of the shape feature is defined as follows. We assume a pre-defined local coordinate frame associated with the subdivision of the original surface (say the XYZ frame). Let the feature instance be defined in a local coordinate frame X'Y'Z'. Without loss of generality, we can assume that the Z'-axis lies along the axis of the approximating cone. We use 5 parameters to define the transformation from XYZ to X'Y'Z' (three coordinates define the shift between the origins, and two independent angles of relative rotation, say, α and β). The cone point lies at some distance, d, from the origin along the Z'-axis. Let the cone half angle be denoted as γ. Finally, we need two parameters, s and t, to identify the left and right directrices defining the boundary of the cone. This fully defines the cone, and each of the two tangent planes along the shared boundaries. Therefore a total of 9 parameters are required for a regression fit between an instance of the shape feature and the underlying geometry. We use an additional 8 parameters to complete the geometric description of the feature: 4 parameters identify the four corner points of the conical patch (for each pair of points that lies along the same directrix, it is sufficient to specify the start and end distance along the directrix from the cone point). Another four scalars identify the four corner points of the planar faces (since the tangential boundary curve is continuous, we only need to specify the length of the boundary edge to locate each corner point).

The objective function of the optimization model is composed of two components:

The first component is made up of the regression terms, measuring the distance of each sampled point on the original surface to the feature. In order to avoid discontinuity, the sampled points are selected in a manner to avoid any region in the neighborhood of the path boundaries. Let $F_i = \{P_{1i}, C_i, P_{2i}\}$ denote the feature instance approximating the i-th panel using a set of m points for the regression. Then we pre-assign some m_1 points to lie close to the planar surface P_{1i},

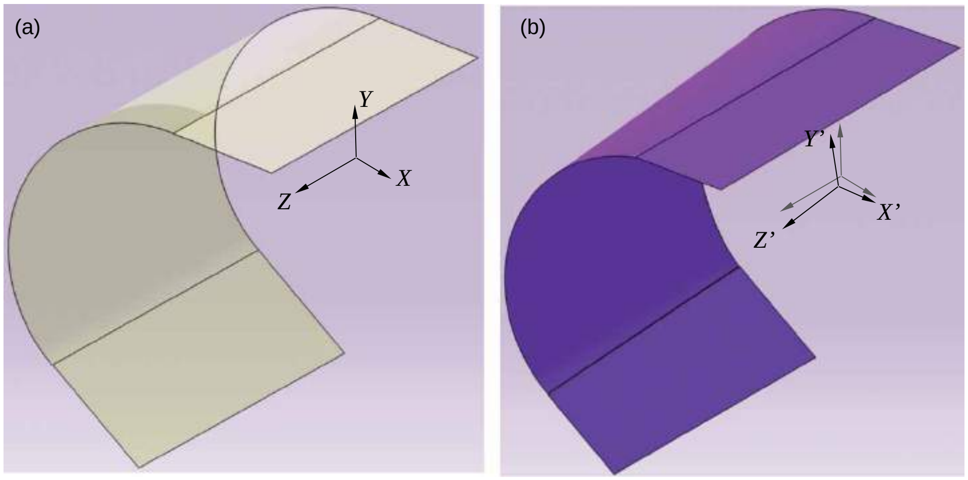

Fig. 8: (a) A segment of the ideal surface patches from the lip region of the cruise terminal; the patch is made up of a free form surface blended onto two planar pieces, one on each end. (b) The developable shape feature used to approximate the surface patch is composed of a set of three patches, a plane, a cone and a plane.

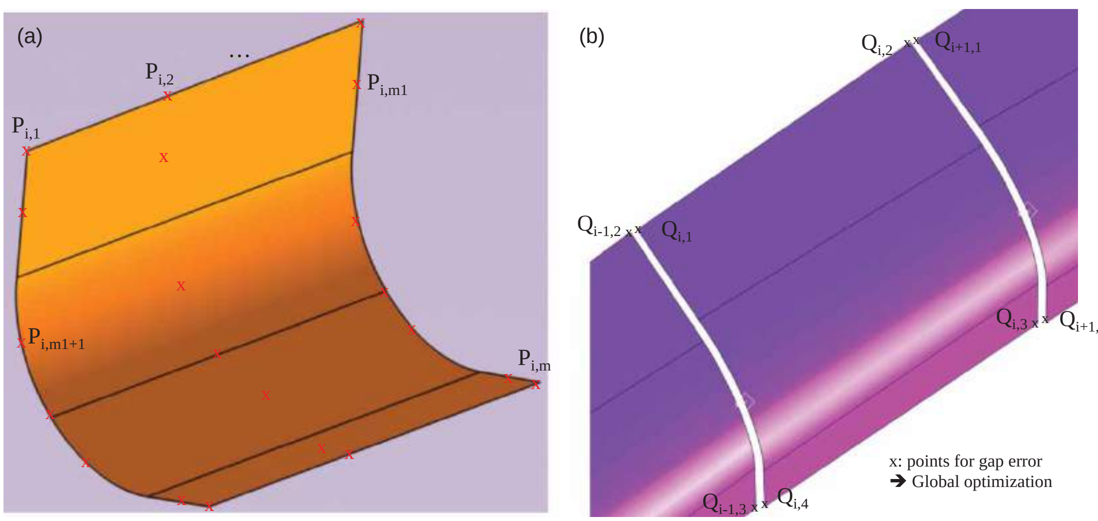

Fig. 9: (a) The set of m points on the reference surface that are used for the surface fitting (regression) portion of the optimization objective function and (b) the set of four points that are used to quantify the inter panel, or gap error.

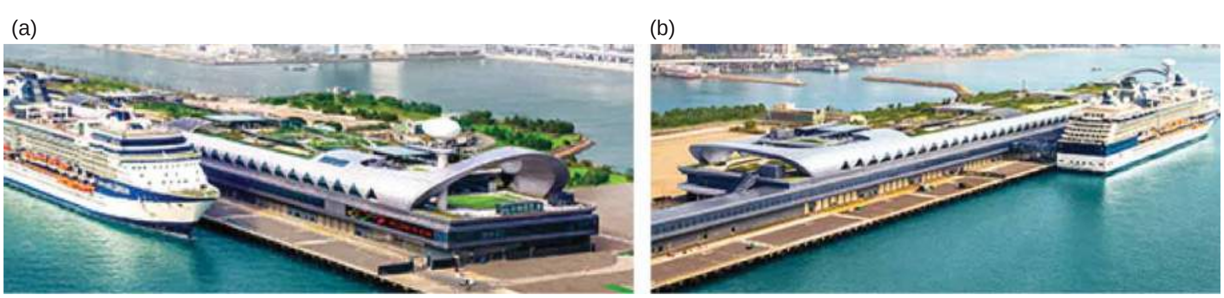

Fig. 10: Kai Tak Cruise terminal showing the: (a) SE lip, (b) NW lip. Source: Hong Kong Tourism Board, http://www.discoverhongkong.com

m_2 points to lie close to the cone C_i, and m_3 points to lie closest to P_{i2} ($m = m_1 + m_2 + m_3$). The formulae for the distance terms, $d_{i,j}$ therefore are expressions for point-plane distance for $1 < j \leq m_1$ and $(m_1 + m_2) < j \leq m$, and the point-cone distance expression for $m_1 < j \leq (m_1 + m_2)$.

The second expression is a set of terms that controls the distance between two neighboring panels at some specific points. For our case study, we use the four corner points of the panels, as indicated in Fig. 9. For this example, the weighted Euclidean distance between the corresponding pairs of points can be used as the error metric. Although this metric was sufficient for our example, the approach can be easily adapted to use other inter-panel error measures.

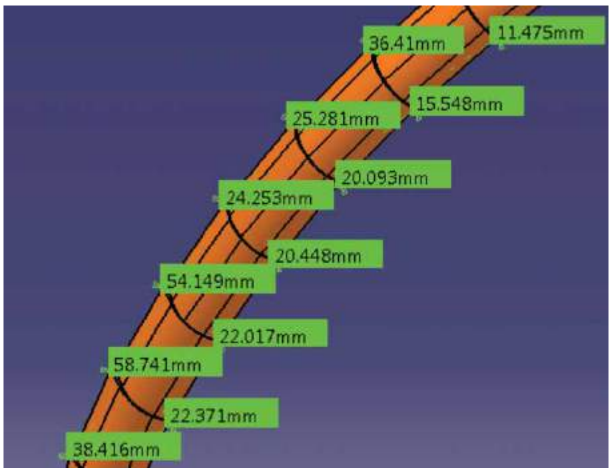

Fig. 11: The maximum gap in the plane-cylinder-plane setting is near 59 mm in NW side.

Therefore our surface fitting problem for a sequence of n panels can be written as:

$$\text{Minimize E} = \sum_{i=1}^{n}\Big[w_j \left\|P_{i,j} - P_{f(j),n(j)}\right\|^2 + w_j dist(P_{i,j},\ ref\ (i))^2\Big] \tag{3}$$

where the operators $f(\mathrm{j})$, $n(\mathrm{j})$ determine the matching between the point $P_{i,j}$ and its appropriate panel and corresponding neighboring point on that panel for the gap error metric, w_{j} are the weights associated with each error term, and $ref(j)$ is the reference surface of feature instance i.

The problem (3) is non-linear and non-convex. We use a simulated annealing style heuristic approach to solve it, as outlined in the pseudocode below. In each stage of the algorithm, a random solution is generated, and then an adaptive stepping steepest descent is used to locate the neighboring local stationary point.

Algorithm: Concurrent Fitting

Input: set of reference surfaces, feature descriptions, acceptable total error E_0, convergence tolerance τ

Output: an instance of a shape feature for each reference surface located in the global coordinate frame

1. x_k = initial solution computed by fitting each feature instance independently
2. δ_k = initial step size
3. $x_{k+1} = x_k - \delta_k \nabla E\,(x_k)$
4. if $\left\|x_k - x_{k+1}\right\| < \tau$ then
5. if $E < E_0$ then
6. report incumbent solution
7. else perturb x_k ; go to step 2
8. else adaptively scale δ_k; go to step 3.

We mention here that the perturbation to the initial (local fitting) solution may be achieved in several possible ways. In our case, our approach is to merely add small random variations to some of the parameters of the feature instances. In the next section, we report the results of using this approach for our case study problem.

4. IMPLEMENTATION AND RESULTS

The approach described above was used to fit plane-cone-plane panel instances to three sets of 8∼10 panels each along the most twisting and curving lip regions of the Kai Tak Cruise terminal building. The building has a lip on each end (the south-east end, SE, and the north-west end, NW). Fig. 11 shows the images of SE and NW side of the Terminal. The North corner of the NW lip included 8 panels, while the South and East corners of the SW lip included 9 and 10 panels respectively. The optimization model was

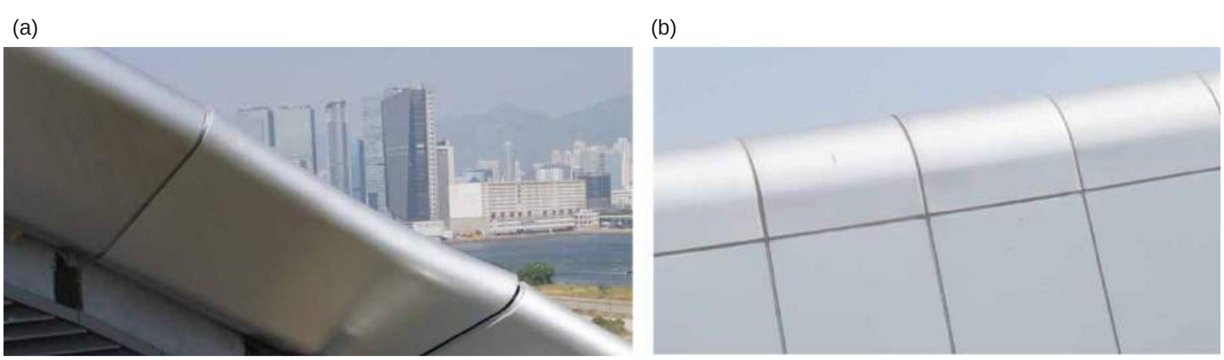

Fig. 12: (a) Panels with large inter-panel gap are not aesthetic, as compared to (b) panels with a optimized inter-panel gaps.

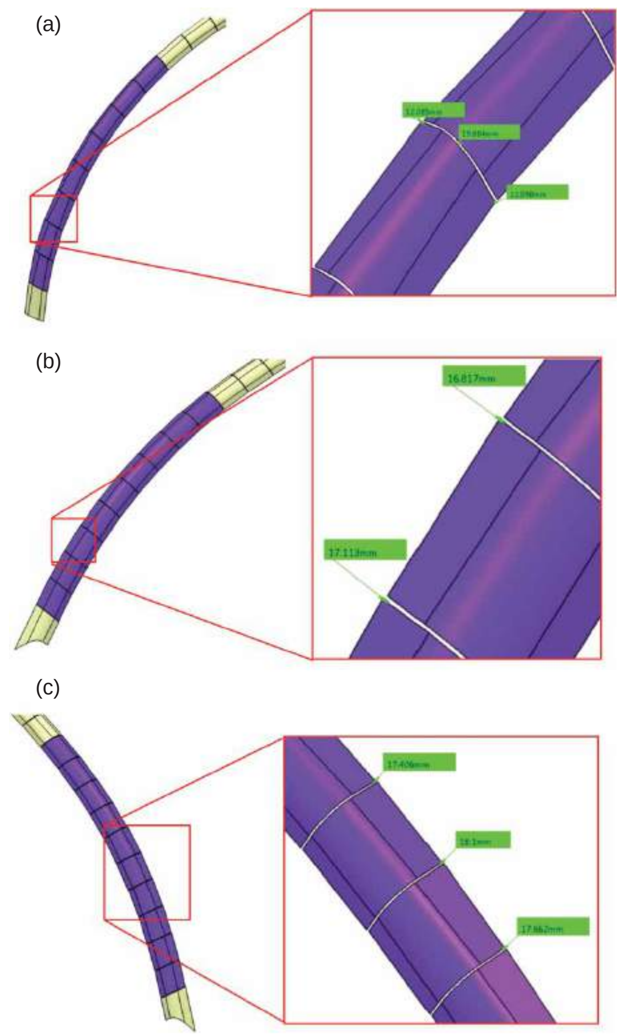

Fig. 13: Final results of plane-cone-plane optimized fitting surface: (a) North side of NW lip (b) South side of SE lip. (c) East side of SE lip.

implemented as an API integrated into a commercial CAD system, CATIA (which was used to model the feature instances as well as the original surface). Our benchmark was the result of a local fitting of panels using the 2-stage approach. The panel models by using the 2-stage process on the same 8 panels on the NW lip are shown in Fig. 11. It can be seen that the outcome of this approach yields inter-panel gaps of up to 58.7 mm. Fig. 12 shows two images from the actual building, one where the inter-panel gaps are of this order of magnitude, and the second one from the lip region where the inter panel gap has been reduced to under 20 mm. Fig. 13 shows the result of our approach for the three sets of panels.

Table 1 summarizes the maximum inter-panel gap for all the panels in our three optimization experiments. The maximum gap in each of these cases when using the 2-stage process are also shown for comparison, and it is clear from this data that our proposed approach yields significantly better results in all cases. The run-time for our optimization models is of the order of ~ 10 minutes for each case.

5. DISCUSSION

We introduced the problem of rationalization of curved panels for architectural skins. We showed that limitations of manufacturing machines limit the type of panel shapes that may be economically fabricated to developable shapes using analytical surfaces such as cylinders or cones. The notion of a shape feature was introduced to represent various varieties of rationalized panels. To our knowledge, all past work on producing rationalized surface panels uses a 2-stage approach, where the first stage performs a local (panel-by-panel) fitting, and the second stage performs the optimized location. We introduce a simple approach that combined the two optimization problems into a single global optimization problem. For moderate sized problems (i.e. involving up to a few hundred variables), this approach was demonstrated to be more effective and to yield better results that the traditional 2-stage approach. Our approach was tested for a particular shape of panels in the context of our case study. Nevertheless, the method can be applied to sets of panels, each defined by arbitrary

Inter-Panel index	SE, South (2-stage)	SE, South (our method)	SE, East (2-stage)	SE, East (our method)	NW (2-stage)	NW (our method)
1	0	6.8	0	9.4	32	11.9
2	32.7	12.7	31.136	11.8	38.4	13.8
3	36.9	17.1	35	17.7	58.7	19.9
4	34.4	16.8	32.5	18.1	54.1	12.4
5	34.5	16.8	31.5	17.4	24.3	16
6	32.9	16.5	30	16.2	25.3	18.2
7	30.5	15	28.6	16.3	36.4	15.5
8	28.01	13.8	26	15.6	27.2	12.3
9	25.533	12	22.1	15	16.8	12.4
10	21.86	16.4	17.8	15.5	–	–

Tab. 1: Maximum gap (mm) comparison between the 2-stage fitting approach and our approach.

user-defined shape features. A possible interesting future research direction is to classify shape features and grid types based on the type of optimization formulation they induce. In particular, problems that are defined in terms of, say, convex optimization could be solved by very robust algorithms even for fairly large problem sizes.

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