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Quadratic log-aesthetic curves

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ABSTRACT
This paper proposes quadratic log-aesthetic curves that are curves whose logarithmic curvature graphs are quadratic. In previous work, generalized log-aesthetic curves are derived by shifting either the curvature or the radius of curvature of log-aesthetic curves. Quadratic log-aesthetic curves are another generalization of log-aesthetic curves by making logarithmic curvature graphs quadratic. We derive the equations of quadratic log-aesthetic curves and clarify their characteristics. For drawing quadratic log-aesthetic curves, we need to compute the inverses of the error and imaginary error functions. We present a method for computing these inverses and confirm that the curves can be generated in real time.

KEYWORDS
Quadratic log-aesthetic curve; logarithmic curvature graph; error and imaginary error functions

1. Introduction

In the design of highly aesthetic surfaces, such as the exterior of automobiles and others, the use of highly aesthetic curves is important since such surfaces are constructed based on aesthetic feature curves. If the curvature of a feature curve is not monotonically varying, the reflected image of the generated surface gets easily distorted, which is not aesthetically pleasing. See Fig. 9 of [17] for example. In aesthetic shape design, the use of curves with monotonically varying curvature (and placing curvature extrema intentionally) is important. Freeform curves, such as Bézier curves or NURBS curves are widely used in CAD systems, but controlling the curvature of freeform curves is not easy. Among various curves with monotonically varying curvature, Harada et al. has proposed log-aesthetic curves based on their original analysis [2] of many aesthetic curves including feature curves of automobiles.

Log-aesthetic curves [2,7,12] are curves with linear logarithmic curvature graphs whose slope is α. Special cases of log-aesthetic curves are the Clothoid if $\alpha = -1$, Nielsen's spiral if $\alpha = 0$, a logarithmic spiral if $\alpha = 1$, the circle involute if $\alpha = 2$, and a circle if $\alpha = \pm\infty$. Various work has been done for log-aesthetic curves. Some of them are: compound-rhythm log-aesthetic curves [13], extension to space curves using the linearity in the logarithmic torsion graph [14], proving the uniqueness of the curve segment when $\alpha < 0$ (curves with inflection points) [6], proving the evolutes of log-aesthetic curves are also log-aesthetic curves [16], fast computation of curve segment using incomplete Gamma function[11], generalization by shifting either the curvature or the radius of curvature[1,10], G^2 Hermite interpolation using three connected segments [9], and application to generating surfaces [3,4,8].

This paper proposes quadratic log-aesthetic curves, which are curves whose logarithmic curvature graphs [15] are quadratic. In previous work, generalized log-aesthetic curves [1,10] have been proposed by Miura and Gobithaasan. Generalized log-aesthetic curves are derived by shifting either the curvature or the radius of curvature of log-aesthetic curves. Quadratic log-aesthetic curves are another generalization of log-aesthetic curves by making logarithmic curvature graphs quadratic. It has been shown that the incomplete Gamma function arise in the tangential angle formulation of log-aesthetic curves [10]. We show that other special functions, which are the error and imaginary error functions, arises in the formulation of quadratic log-aesthetic curves.

We derive the equations of quadratic log-aesthetic curves and clarify their characteristics. One notable difference from generalized log-aesthetic curves is that quadratic log-aesthetic curves include curves with finite arc lengths and their curvature varying from 0 to infinity. We show that such curves can be obtained if the quadratic coefficient γ in the logarithmic curvature graph is negative. For drawing quadratic log-aesthetic curves, we need to compute the inverses of the error and imaginary error

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functions. We present a method for computing these inverses and confirm that the curves can be generated in real time.

2. Error and imaginary error functions

This section briefly reviews the error and imaginary error functions [5]. As will be shown later, we have to compute these functions as well as their inverse functions for drawing quadratic log-aesthetic curves. Since we compute these functions using floating point arithmetic, such as the double precision binary floating point arithmetic, we have to be careful about the domain of these functions.

The error function $\mathrm{erf}(z)$ and imaginary error function $\mathrm{erfi}(z)$ are defined as

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, \mathrm{d}t, \tag{2.1}$$

$$\mathrm{erfi}(z) = -i\,\mathrm{erf}(iz) \tag{2.2}$$

where i is the imaginary unit. Although i appears in the right side of Eqn. (2.2), $\mathrm{erfi}(z)$ is always real if z is real. Figure 1 shows the error and imaginary error functions.

When we compute $\mathrm{erf}(z)$ or $\mathrm{erfi}(z)$, we have to be careful that we are computing using the double precision. As $|z|$ approaches 2, $\mathrm{erf}(z)$ approaches ± 1. In the double precision, if $|z|$ is greater than approximately 5.92, $\mathrm{erf}(z)$ is becomes essentially 1. Thus we safely assume

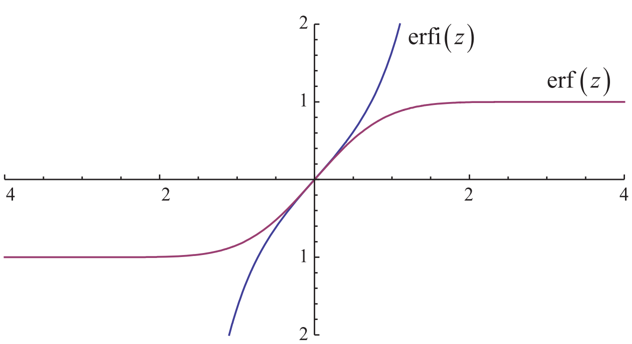


Figure 1. The error and imaginary error functions.

$|z| \leq 5$ when computing $\mathrm{erf}(z)$. When computing $\mathrm{erfi}(z)$, we have to be careful so that $|\mathrm{erfi}(z)|$ is within the range of the double precision. We assume $|z| \leq 26$ when computing $\mathrm{erfi}(z)$, since $\mathrm{erfi}(26)(\approx 8.31464 \times 10^{291})$ is within the range of the double precision. These assumptions are used as the bounds for computing inverses of these functions, which are necessary for drawing quadratic log-aesthetic curves.

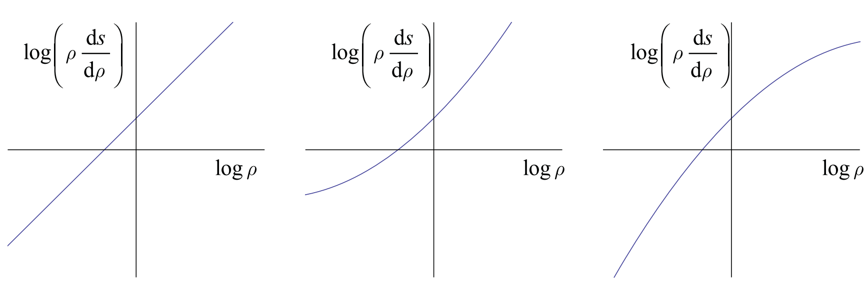
3. The fundamental equation of quadratic log-aesthetic curves

The equations of log-aesthetic curves are derived from the linearity in logarithmic curvature graphs. Similarly, the equation of quadratic log-aesthetic curves are derived from quadratic curves in logarithmic curvature graphs. Logarithmic curvature graphs are graphs whose horizontal axis is $\log \rho$ and vertical axis is $\log(\rho \mathrm{d}s/\mathrm{d}\rho)$. Here, ρ is the radius of curvature and s is the arc length. Fig. 2(a) shows the linear logarithmic curvature graph with its slope $\alpha = 1$. See [2,7,11,13] for more details of log-aesthetic curves and logarithmic curvature graphs.

Quadratic log-aesthetic curves are curves whose logarithmic curvature graphs are quadratic as shown in Fig. 2 (b) or (c). Quadratic curves includes ellipses (including circles), parabolas, and hyperbolas. Among them, only parabolas in the logarithmic curvature graphs guarantees the monotonicity of the curvature since other types of quadratic curves may have a turn in the horizontal axis ($\log \rho$). A parabola in the logarithmic curvature graph is

$$\log\left(\rho \frac{\mathrm{d}s}{\mathrm{d}\rho}\right) = \gamma(\log \rho)^2 + \alpha \log \rho + c \tag{3.1}$$

where γ, α are quadratic and linear coefficients respectively and c is a constant. Eqn. (3.1) is the fundamental equation for quadratic log-aesthetic curves. Note that Eqn. (3.1) includes a line if $\gamma = 0$, which means that Eqn. (3.1) is the generalization of the fundamental equation



(a) $\alpha = 1, \gamma = 0$ (b) $\alpha = 1, \gamma = 0.02$ (c) $\alpha = 1, \gamma = -0.02$

Figure 2. Linear and quadratic logarithmic curvature graphs.

of log-aesthetic curves. In comparison with log-aesthetic curves, quadratic log-aesthetic curves have additional parameter γ. If we modify γ near 0, we can generate curves close to log-aesthetic curves guaranteeing the monotonicity of the curvature. If $\gamma = 0$, quadratic log-aesthetic curves become exactly log-aesthetic curves.

4. Deriving the equations for quadratic log-aesthetic curves

This section derives the curvature κ of quadratic log-aesthetic curves as a function of arc length s based on Eqn. (3.1). Once the curvature function is derived the curve can be drawn using Frenet–Serret formulas. Using curvature κ, Eqn. (3.1) can be written as

$$\log\left(-\kappa\frac{\mathrm{d}s}{\mathrm{d}\kappa}\right) = \gamma(\log\kappa)^2 - \alpha\log\kappa + d. \tag{4.1}$$

Eqn. (3.1) and Eqn. (4.1) are essentially the same equations. Taking the exponential of both sides of Eqn. (4.1), we get

$$\frac{\mathrm{d}s}{\mathrm{d}\kappa} = -\kappa^{-\alpha+\gamma\log\kappa-1}e^{d}. \tag{4.2}$$

Let

$$\Lambda = e^{-d}.$$

The reason that Λ is set to like this is that the curve becomes a circular arc if $\Lambda = 0$ [11]. Then Eqn. (4.2) becomes

$$\frac{\mathrm{d}s}{\mathrm{d}\kappa} = -\frac{\kappa^{-\alpha+\gamma\log\kappa-1}}{\Lambda}. \tag{4.3}$$

Integrating Eqn. (4.3) with respect to κ, we get

$$s(\kappa) = \begin{cases} -\dfrac{e^{-\frac{\alpha^2}{4\gamma}}\sqrt{\pi}\,\mathrm{erfi}\left(\frac{-\alpha+2\gamma\log\kappa}{2\sqrt{\gamma}}\right)}{2\sqrt{\gamma}\Lambda} & \text{if } \gamma > 0 \\ -\dfrac{e^{-\frac{\alpha^2}{4\gamma}}\sqrt{\pi}\,\mathrm{erf}\left(\frac{\alpha-2\gamma\log\kappa}{2\sqrt{-\gamma}}\right)}{2\sqrt{-\gamma}\Lambda} & \text{if } \gamma < 0 \\ \dfrac{\kappa^{-\alpha}}{\alpha\Lambda} & \text{if } \alpha \neq 0 \text{ and } \gamma = 0 \\ -\dfrac{\log\kappa}{\Lambda} & \text{if} \alpha = 0 \text{ and } \gamma = 0. \end{cases} \tag{4.4}$$

Similarly as in [11], we consider the standard form. In the standard form, we set $\kappa = 1$ at $s = 0$. Then Eqn. (4.4) becomes

$$s(\kappa) = \begin{cases} \dfrac{-e^{-\frac{\alpha^2}{4\gamma}}\sqrt{\pi}\left(\mathrm{erfi}\left(\frac{-\alpha+2\gamma\log\kappa}{2\sqrt{\gamma}}\right) - \mathrm{erfi}\left(\frac{-\alpha}{2\sqrt{\gamma}}\right)\right)}{2\sqrt{\gamma}\Lambda} \\ \quad \text{if } \gamma > 0 \\ \dfrac{-e^{-\frac{\alpha^2}{4\gamma}}\sqrt{\pi}\left(\mathrm{erf}\left(\frac{\alpha-2\gamma\log\kappa}{2\sqrt{-\gamma}}\right) - \mathrm{erf}\left(\frac{\alpha}{2\sqrt{-\gamma}}\right)\right)}{2\sqrt{-\gamma}\Lambda} \\ \quad \text{if } \gamma < 0 \\ \dfrac{\kappa^{-\alpha}-1}{\alpha\Lambda} \quad \text{if } \alpha \neq 0 \quad \text{and} \quad \gamma = 0 \\ -\dfrac{\log\kappa}{\Lambda} \quad \text{if } \alpha = 0 \quad \text{and} \quad \gamma = 0. \end{cases} \tag{4.5}$$

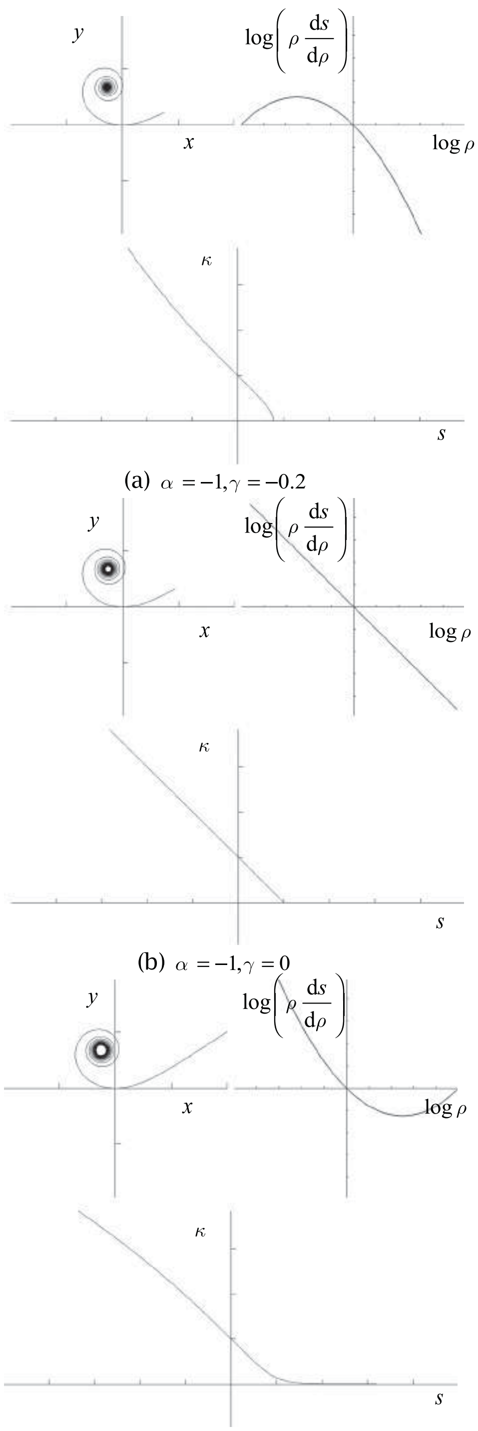
Solving $s(\kappa)$ for κ, we get

$$\kappa(s) = \begin{cases} e^{\frac{\alpha+2\sqrt{\gamma}\,\mathrm{erfi}^{-1}\left(-\frac{2\sqrt{\gamma}e^{\frac{\alpha^2}{4\gamma}}\Lambda s+\sqrt{\pi}\,\mathrm{erfi}\left(\frac{\alpha}{2\sqrt{\gamma}}\right)}{\sqrt{\pi}}\right)}{2\gamma}} \\ \quad \text{if } \gamma > 0 \\ e^{\frac{\alpha-2\sqrt{-\gamma}\,\mathrm{erf}^{-1}\left(\frac{-2\sqrt{-\gamma}e^{\frac{\alpha^2}{4\gamma}}\Lambda s+\sqrt{\pi}\,\mathrm{erf}\left(\frac{\alpha}{2\sqrt{-\gamma}}\right)}{\sqrt{\pi}}\right)}{2\gamma}} \\ \quad \text{if } \gamma < 0 \\ (1+\alpha\Lambda s)^{-\frac{1}{\alpha}} \quad \text{if } \alpha \neq 0 \quad \text{and} \quad \gamma = 0 \\ e^{-\Lambda s} \quad \text{if } \alpha = 0 \quad \text{and} \quad \gamma = 0. \end{cases} \tag{4.6}$$

In Eqn. (4.5) and (4.6), the cases of $\gamma = 0$ are exactly the same as the ones in log-aesthetic curves. If $\gamma > 0$ or $\gamma < 0$, inverse imaginary function $\mathrm{erfi}^{-1}(x)$ or inverse error function $\mathrm{erf}^{-1}(x)$ arises. For computing the inverse of these functions, we use a hybrid method combining the bisection method and the Newton's method. In the region where $|z|$ in Eqn. (2.1) or Eqn. (2.2) is near 0 where the Newton's method is reliable, we use the Newton's method. Otherwise, the bisection method is used. In the bisection method, the lower and upper bounds of $\mathrm{erf}(z)$ are -5 and 5 respectively, and the lower and the upper bound of $\mathrm{erfi}(z)$ are -26 and 26 respectively. These bounds are derived as was described in Section 2.

Since the curvature of a quadratic log-aesthetic curve varies from 0 to ∞, we have to be careful about the bounds of arc length s. s may have upper and lower bounds depending on γ, α, and Λ. Putting $\kappa = 0$ and $\kappa = \infty$ into Eqn. (4.5), we get

$$s(0) = \begin{cases} \infty & \text{if } \gamma > 0 \\ \dfrac{e^{-\frac{\alpha^2}{4\gamma}}\sqrt{\pi}\left(1+\operatorname{erf}\left(\frac{\alpha}{2\sqrt{-\gamma}}\right)\right)}{2\sqrt{-\gamma}\Lambda} & \text{if } \gamma < 0 \\ \infty & \text{if } \alpha \geq 0 \text{ and } \gamma = 0 \\ -\dfrac{1}{\alpha\Lambda} & \text{if } \alpha < 0 \text{ and } \gamma = 0 \end{cases} \tag{4.7}$$



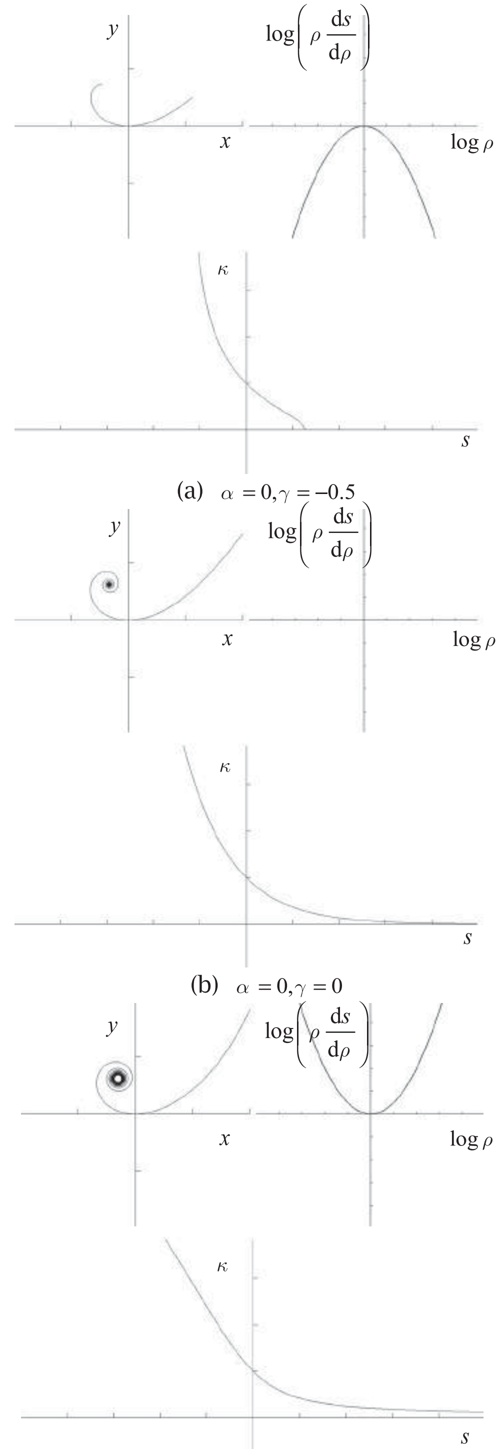
(a) $\alpha = -1, \gamma = -0.2$

(b) $\alpha = -1, \gamma = 0$

(c) $\alpha = -1, \gamma = 0.2$

Figure 3. $\alpha = -1$ with different γ.

$$s(\infty) = \begin{cases} -\infty & \text{if } \gamma > 0 \\ \dfrac{e^{-\frac{\alpha^2}{4\gamma}}\sqrt{\pi}\left(-1+\operatorname{erf}\left(\frac{\alpha}{2\sqrt{-\gamma}}\right)\right)}{2\sqrt{-\gamma}\Lambda} & \text{if } \gamma < 0 \\ -\dfrac{1}{\alpha\Lambda} & \text{if } \alpha > 0 \text{ and } \gamma = 0 \\ -\infty & \text{if } \alpha \leq 0 \text{ and } \gamma = 0. \end{cases} \tag{4.8}$$



(a) $\alpha = 0, \gamma = -0.5$

(b) $\alpha = 0, \gamma = 0$

(c) $\alpha = 0, \gamma = 0.5$

Figure 4. $\alpha = 0$ with different γ.

$s(0)$ and $s(\infty)$ are the upper and lower bounds of arc length s, respectively. These bounds are necessary for drawing quadratic log-aesthetic curves. If $s(0) = \infty$, it means that the arc length to the point at $\kappa = 0$ is infinity. Thus the inflection point does not exist. In other words, the inflection point is at infinity. Similarly, if $s(\infty) = \infty$, the arc length to the point at $\kappa = \infty$ is infinity. In summary, the following properties can be derived for quadratic log-aesthetic curves.

(1) The curve includes an inflection point if $\alpha < 0$ and $\gamma = 0$ or if $\gamma < 0$.

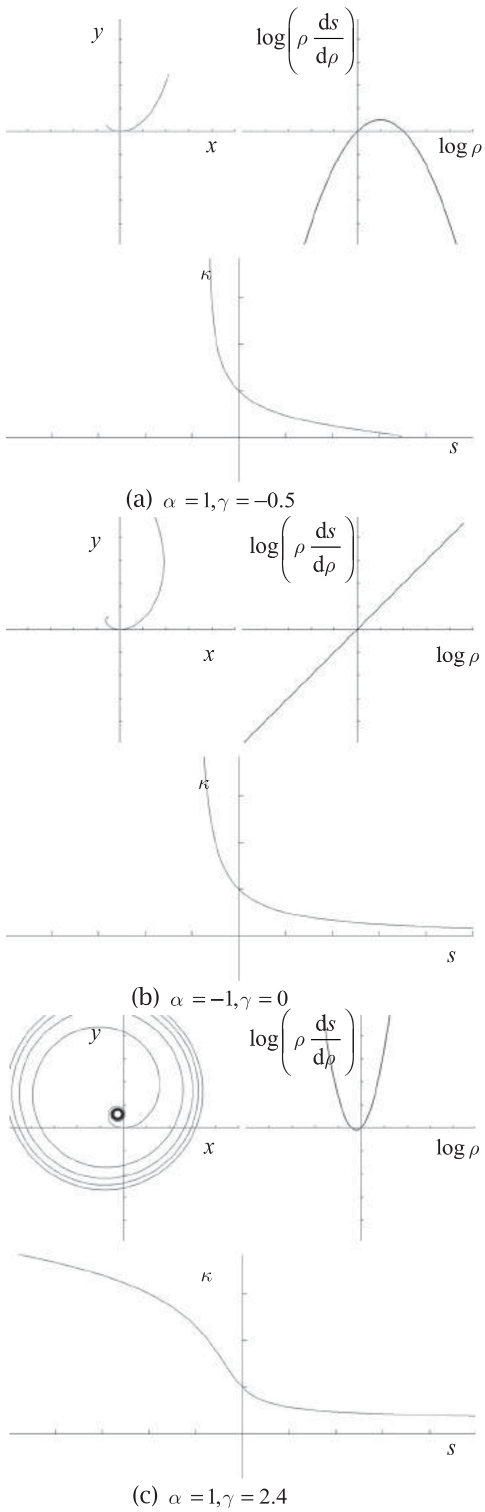


Figure 5. $\alpha = 1$ with different γ.

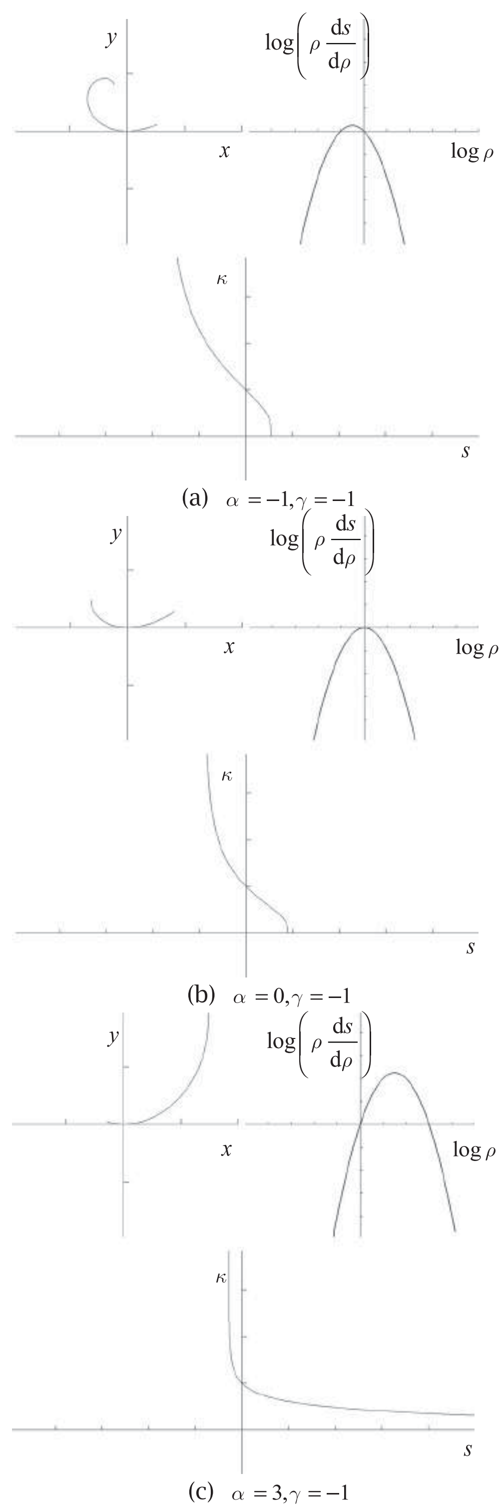


Figure 6. $\gamma = -1$ with different α.

(2) The curve includes a point at $\kappa = \infty$ if $\alpha > 0$ and $\gamma = 0$ or if $\gamma < 0$.
(3) If $\gamma < 0$, the arc length of the curve is finite and its curvature varies from 0 to ∞ without depending on the value of α.
(4) For curves with $\gamma < 0$, the arc length gets longer as Λ gets smaller.

Property (4) can be verified from Eqn. (4.7) and (4.8). $\left(1 + \mathrm{erf}\left(\alpha / \left(2\sqrt{-\gamma}\right)\right)\right)$ in the numerator of $\gamma < 0$ in Eqn. (4.7) is always positive since $\mathrm{erf}(z) \in (0, 1)$. Thus $s(0)$ gets larger as Λ gets smaller, since the numerator of $\gamma < 0$ in Eqn. (4.7) are positive. Similarly, since the numerator of $\gamma < 0$ in Eqn. (4.8) are negative, $s(\infty)$ gets smaller as Λ gets smaller. Thus the arc length of the curve with $\gamma < 0$, which is $s(0) - s(\infty)$, gets longer as Λ gets smaller. In the limit of $\Lambda = 0$, the arc length of the curve becomes infinite with its curvature being a constant.

Once Eqn. (4.6) and the bounds of arc length are derived, the curve can be drawn by integrating the Frenet–Serret formulas with an initial condition. Similarly as in [5], we set the tangent and the curve point at $s = 0$ as $\left[\begin{array}{cc} 1 & 0 \end{array}\right]^{\mathrm{T}}$ and the origin, respectively.

5. Results

We have implemented the code for drawing a quadratic log-aesthetic curve including its logarithmic curvature graph and curvature plot using C++ on an Intel Core i7 2.2 GHz processor. In our experimental system, the curve and its related graphs are shown in real time by interactively modifying the values of γ, α and Λ.

Fig. 3, 4, 5 and 6 show quadratic log-aesthetic curves with various α, γ and their logarithmic curvature graphs and curvature plots. In all of Fig. 3, 4, 5 and 6, Λ is set to 1. In each of (a) to (c) of these figures, the upper left figure is a quadratic log-aesthetic curve, the upper right is the logarithmic curvature graph, and the bottom is the curvature plot. 3, 4 and 5 are the curves of $\alpha = -1, 0, 1$, respectively, with various γ. As shown in Eqn. (4.4) and (4.5), the arc length of the quadratic log-aesthetic curve is finite if $\gamma < 0$ with its curvature varying from 0 to infinity. These cases are shown in Fig. 3(a), 4(a), 5(a) and 6(a). If $\gamma = 0$, quadratic log-aesthetic curves become exactly log-aesthetic curves. If $\gamma > 0$, the arc length of the curve become infinite (Fig. 3 (c), for example). In other words, the points at $\kappa = 0$ and $\kappa = \infty$ are at infinity. Fig. 7

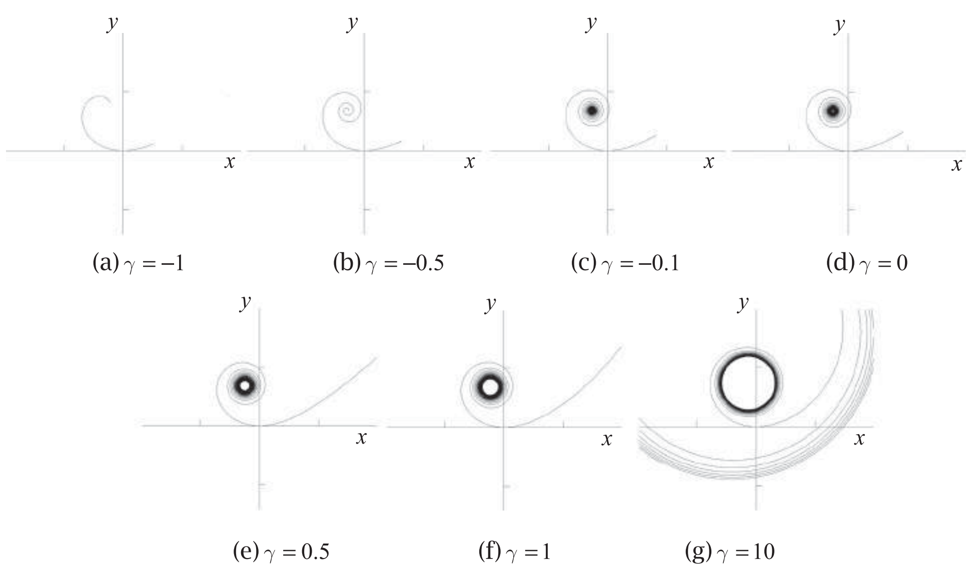


Figure 7. $\alpha = -1$ with various γ ($\Lambda = 1$).

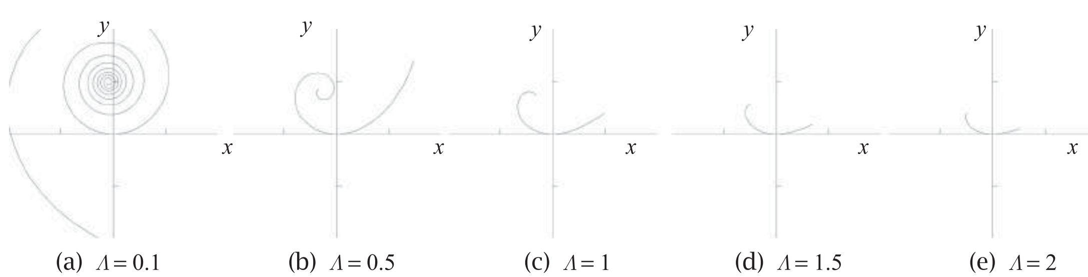


Figure 8. $\alpha = -0.2, \gamma = -0.5$.

shows how a log-aesthetic curve ($\alpha = -1$) changes by modifying the quadratic coefficient γ.

Fig. 8 shows the curves of $\alpha = -0.2, \gamma = -0.5$ but with different Λ. As Λ is decreased, the arc length of the curve gets longer if $\gamma < 0$. The shape of the curve also changes if the value of Λ is changed except when $\alpha = 1$ [12]. Fig. 9 shows an interesting example of $\alpha = 1, \gamma = 10, \Lambda = 0.5$. From the curve shape and its curvature plot, the curve gets closer to the circular arc (constant κ) as s gets close to $\pm\infty$. This fact means that G^1 interpolation algorithm of [5] does not work properly since more than one curves that fit the specified control triangle may be found. Thus a different method is necessary and we are working with this problem.

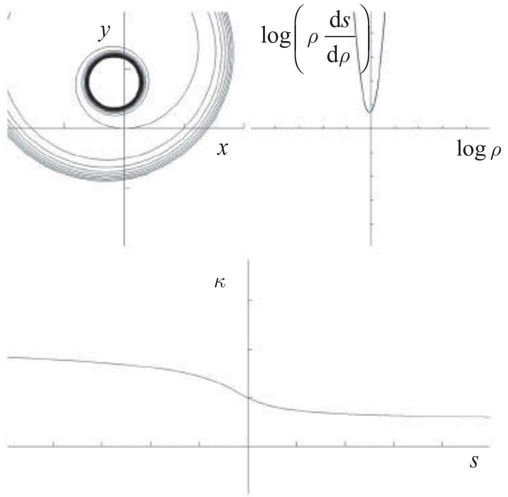


Figure 9. $\alpha = 1, \gamma = 10, \Lambda = 0.5$.

6. Conclusions

This paper proposed quadratic log-aesthetic curves by extending log-aesthetic curves so that the logarithmic curvature graphs becomes quadratic. We derived the curvature function in terms of arc length for drawing the curve and clarified the characteristics. We have implemented the proposed curve in C++ and confirmed that the curves can be generated fully in real time. Quadratic log-aesthetic curves have additional degree of freedom γ and can represent a curve segment with finite arc length and the curvature varying from 0 to ∞ if $\gamma < 0$. Future work includes more detail analysis of the characteristics of the curves, and the application to G^1 and G^2 Hermite interpolations.

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