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Parametric co-design of modular free-form 2-manifolds

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ABSTRACT
A set of modular 2-manifold surface components has been designed that allows the assembly of free-form geometrical sculptures representing single-sided or double-sided surfaces ranging from genus 2 to genus 22. The paper describes the interplay between parameters that define the overall symmetry of the whole structure and other parameters that define the geometry of the individual modular components.

KEYWORDS
Tubular building blocks; fused-deposition modeling; Klein bottles

1. Introduction

For the Exhibition of Mathematical Art at the Joint Mathematics Meeting 2016 [2] I had set myself the goal to create a couple of sculptures (Fig. 1) that could serve as mathematical visualization models for non-orientable surfaces of higher genus, but which would also hold up as aesthetically pleasing free-form sculptures ("Super-Bottles"), in their own right.

Inspired by project LEGO-Knots [4][5], I designed a small set of modular components that could be combined in many different ways to make single-sided ($\sigma = 1$) or double-sided ($\sigma = 2$) 2-manifolds of genus $g = 2$ and higher. The parts should be suitable for being built with Fused Deposition Modeling or with Selective Laser Sintering, and they should result in sculptures 1–2 feet tall.

The starting point for my design was the classical Klein-bottle. The key geometrical feature here is the Klein-bottle-mouth ("KBM") at the top of Figure 2(a), where a thick tube turns outside-in, like a sock being inverted, and the thinner inner tube then emerges through the side-wall of the thicker tube. To allow the composition of surfaces of higher genus, this basic module has to be enhanced to a 3-way junction, where one of the three tubular arms exposes the opposite side of the surface from the one visible at the other two tubular ends (Fig. 2b). This can be achieved in several different ways. I started out by sketching several possible geometries and contemplating which ones would lead to an attractive and relatively compact modular element (Fig. 3). I soon rejected as too "cumbersome" any designs that looked like a combination of two individual KBMs in a single 3-way junction. I also eliminated the possibility of making a regular, non-inverting tubular 3-way junction and then inverting one or two of the tubular ends with a cross-cap-like pinch in the tube (Fig. 2c), or with a split into 3-or more twisted ribbons (Fig. 2d), as used in Roelofs' "Moebiustorus" [3].

Among many possible designs, I focused on the four model geometries shown in Figure 3 to realize an integrated KBM-junction with three arms. I was looking for a few variants that would allow a rich mix of different geometries, but which would still form some harmonious overall sculpture when joined together. These modules all share a toroidal body as a common style element. In (a) the two "thick" branches merge into the outside of the toroidal ring, while a "thin" inverted tube emerges from the central tunnel of the torus. In (b) two thin tubes merge and jointly enter the torus tunnel. In (c) two thin tubes individually penetrate the ring of the torus, merge inside, and join up with the torus tunnel. In (d) the torus has been replaced with its convex hull, and the thick branch emerges along the central axis of this body, while two thin tubes enter separately through the perimeter, merge inside, and exit jointly in a KBM structure at the opposite side. All self-intersections, which are unavoidable when one immerses a Klein-bottle in 3D Euclidean space, are eliminated by cutting suitable openings into the surface; they are made just large enough to let a "thin" tube section pass through. Types (a) and (b) have only a single *puncture*, while (c) and (d) need two punctures.

In order to make these modules fit together in many different ways, all tubular arms terminate with a consistent "standard arm-diameter." Moreover, the overall geometrical structure of the whole assembly is a cubic

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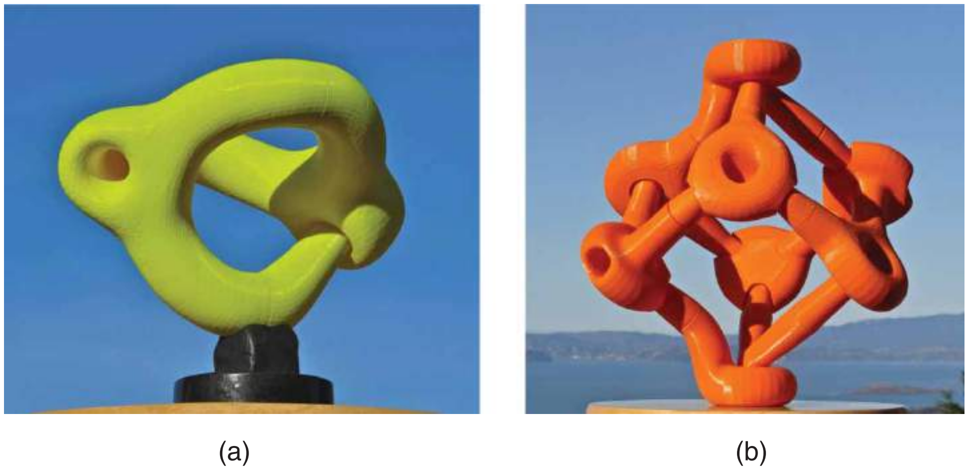


Figure 1. "Super-Bottle" sculptures exhibited at JMM 2016: (a) $\sigma = 1$, g = 4; (b) $\sigma = 1$, g = 10.

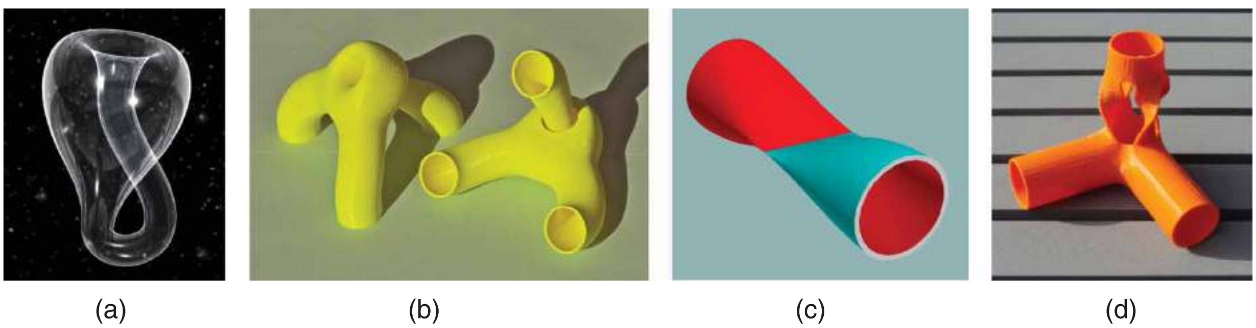


Figure 2. (a) Klein bottle; (b) 3-way KBM-junction; (c) surface-everting kink in pipe; (d) tube eversion by splitting it into four ribbons as in Roelofs' "Moebiustorus" [3].

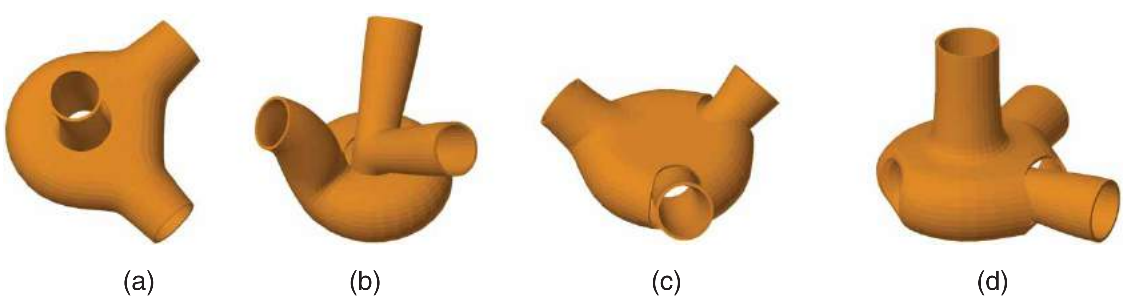


Figure 3. Four different ways of making an integrated KBM-junction.

graph (all nodes are 3-way junctions) based on some regular Platonic polyhedron (tetrahedron, cube, or dodecahedron), or based on a simple hosohedron structure with just two valence-3 junctions. The symmetry of these edge-graphs ensures that all the open tube pairs will join up with proper alignment. Eight modules based on the set shown in Figure 3 readily combine into the edge-frame of a cube, since the angles between any two tubular arms has been made exactly 90°.

The parts outlined in Figure 3 can be combined in hundreds of different ways. They have been designed so that the topological constraints of this project will be satisfied; but this does not guarantee that an aesthetically pleasing sculpture will result. For the latter goal, a complete assembly of eight KBM modules has to be evaluated. Only in this context can one decide what might be the optimal size of the polyhedral edge frame on which the sculpture is based in relation to the exact shapes of the individual modules. In this global setting one would then like to adjust and fine-tune the relative distance between adjacent junction parts as well as the thickness of the connecting tubes between them, i.e., the "standard arm-diameter." All of this should be made possible without losing the modularity and general reconfigurability of the whole module set. Thus each of the modules should adhere to a few global parameters that can be adjusted in this view of the complete sculpture. In addition, each module may have a few individual parameters that can be optimized once the global parameter values have been decided upon.

The design of the KBM modules to be fabricated thus turns out to be an iterative process. Some modules originally designed on an individual basis just did not want to fit into an overall satisfactory sculpture and had to be redesigned significantly or rejected. But the overall sculpture could not be visualized until a few of the individual modules had been designed in sufficient details and with the needed parameterization.

Even for a single object, creating a robust parametrization can be a difficult task. As some parameters exceed their practically allowable range – rather than the range that can be chosen on a given slider – some dependent values may become nonsensical, e.g., producing negative radii for cylinders or spheres. When a system is composed of mutually interacting components that are designed to be combined in many different ways, these problems become even harder. This paper describes some of the geometrical details of this process; it may serve as a guide for other situations, where a modular set of parts has to be designed from scratch with the goal to allow a large number of different compositions.

2. Modularization and parameterization

The geometrical modeling was structured to define for each KBM-junction module a coarse polyhedral mesh of the desired topology and of roughly the right geometry. This mesh is then refined by two or three steps of Catmull-Clark subdivision [1], and is subsequently turned into a material entity by forming an offset surface of appropriate thickness. Many CAD tools have difficulties performing subdivision and/or offsetting operations on single-sided, non-orientable starting meshes. But fortunately all KBM components are well-behaved, orientable 2-manifolds with no self-intersections; the single-sidedness of the surface only emerges when the components are joined together.

The two parts shown in Figure 2(b) can be joined in three different ways; one of them (Fig. 4a) results in a torus of genus 2, while the other two options form the connected sum of two Klein bottles with two punctures and with a non-orientable genus (aka: *demigenus* or *Euler genus*) of $g = 4$ (Fig. 1a).

The parts shown in Figure 3 were designed to readily join together into a tubular cube-frame structure (Fig. 1b), so the three arms have angles of 90° between them. For most of the possible assemblies the result will be a single-sided ($\sigma = 1$) surface of genus 10, corresponding to the connected sum of 5 Klein-bottles with a total of 12 punctures. In a few instances when the “inside” tubes and “outside” tubes are carefully matched up, the result is a two-sided ($\sigma = 2$), orientable 5-hole torus of genus 5 (also with 12 punctures) (Fig. 4b).

To make a straight tetrahedral configuration, four new KBM modules could be fabricated in which the angle between adjacent arms is only 60°. However, a more modular approach is to introduce some additional curved connector parts that bend just the right amount (38.96°) to allow to re-use the 90° cube-corner modules in a tetrahedral configuration (Fig. 4c).

These cube-corner modules can also be combined into less regular polyhedral frames. By adding six curved connector parts that bend through only 30°, six cube-corner modules can be assembled into a 3-sided prism frame of genus $8/\sigma$. By judiciously choosing the orientation of the KBM modules at all corners, the double-sidedness ($\sigma = 2$) of this surface can be maintained (Fig. 4d).

There are four different cube-corner modules to choose from (Fig. 3), and each can be used in three different orientations at the corners of various tubular frame structures. In order to make sure that these components can be used in a modular manner, one has to start with a high-level plan for the whole structure. A regular (or semi-regular) polyhedral wire frame is selected and the mid-points of all its edges are defined as the

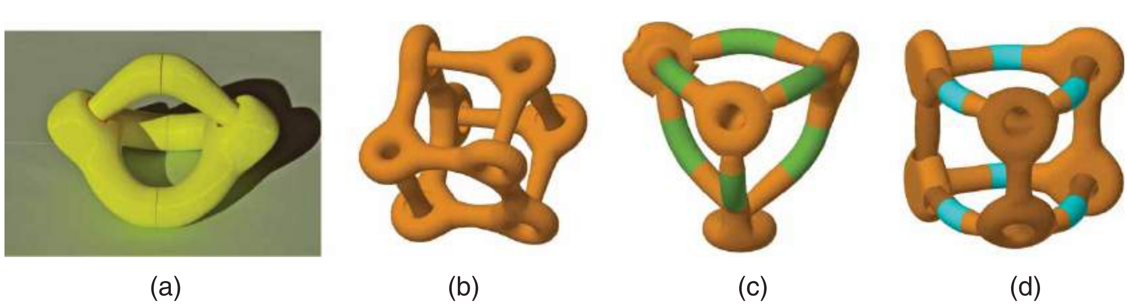


Figure 4. Assemblies of KBM-modules: (a) $\sigma = 2$, $g = 2$; (b) $\sigma = 2$, $g = 5$; (c) $\sigma = 1$, $g = 6$; (d) $\sigma = 2$, $g = 4$.

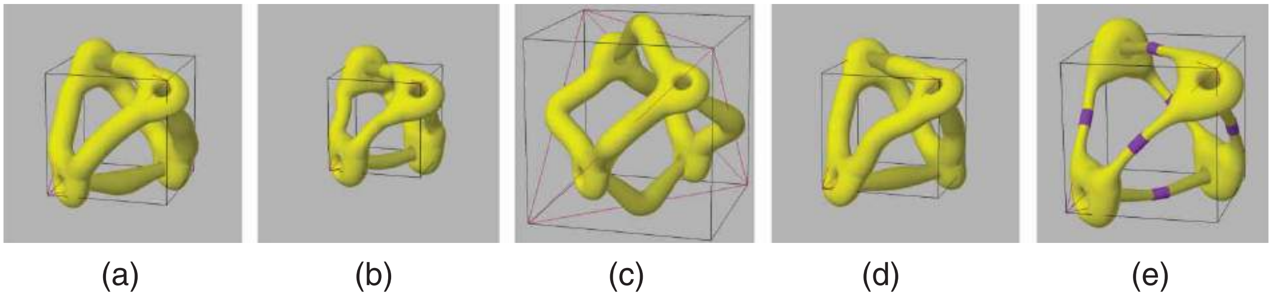


Figure 5. Change of framework: (a) default; (b) frame size; (c) arm bulge.; (d) KBM tilt; (e) arm diameter.

junction points where the tubular arms of adjacent KBM modules will join with a consistent, globally specified arm-diameter.

Once a whole sculpture has been assembled in virtual form, all parameters can be fine-tuned to give the overall most satisfactory result. Figure 5 illustrates the flexibility of the chosen parameterization in the context of a simpler tetrahedral framework. Some parameters control the overall network geometry: they define the overall size of the polyhedral edge frame (Fig. 5b), the bulging of the connecting tubes between the different KBM modules (Fig. 5c), the positioning and tilting of the individual KBM modules (Fig. 5d), and the arm diameter at the tube junctions (Fig. 5e).

3. Core geometry of the KBM module

Other parameters define the geometries of the individual KBM modules. Their shapes are specified as coarse polyhedral meshes, which are then subjected to two or three iterations of Catmull-Clark subdivision [1] in order to obtain a smooth, free-form shape. The defining part of all of these modules is a toroidal body from which several arms emerge in different ways. This main body is modeled as a coarsely tiled toroid, in which the major and minor circles are realized by regular m-gons and n-gons, respectively. The values for m and n may be chosen individually for each type of KBM toroid to allow its arms to emerge most naturally at a desired angle. I experimented with values in the range from 5 to 10; but in most cases I found that $m = n = 8$ was most appropriate for the symmetries and angles involved in these components. The coordinates for the vertices of this toroidal body are generated procedurally with simple explicit formulas, which are parameterized with the sizes of the two radii and the values m and n for the discretization of the two circles. Some of the facets of this toroidal body are eliminated to create openings with rims from which some arms may grow, or to form larger openings in the toroidal wall, through which a thinner arm may pass from the inside to the outside (Fig. 6). Specifically, eliminating three contiguous quadrilateral facets along the minor circle of the toroid leaves an 8-sided opening, which then yields a good match for attaching an arm with an octagonal cross section.

The tubular arms of the individual KBM modules are realized as progressive sweeps using an octagonal cross section. A progressive sweep most easily allows to form a flexible connection between the constraints imposed by the (semi-)regular polyhedral frame that defines the overall structure of the assembly and the detailed core geometry of an individual KBM module, which may be scaled and placed in varying orientations. The octagonal cross section also offers a good match for the symmetries of the polyhedral frames considered. It also would allow to make integrated, 4-fingered, "gender-neutral" connectors at the ends of the tubular arms (see next Section).

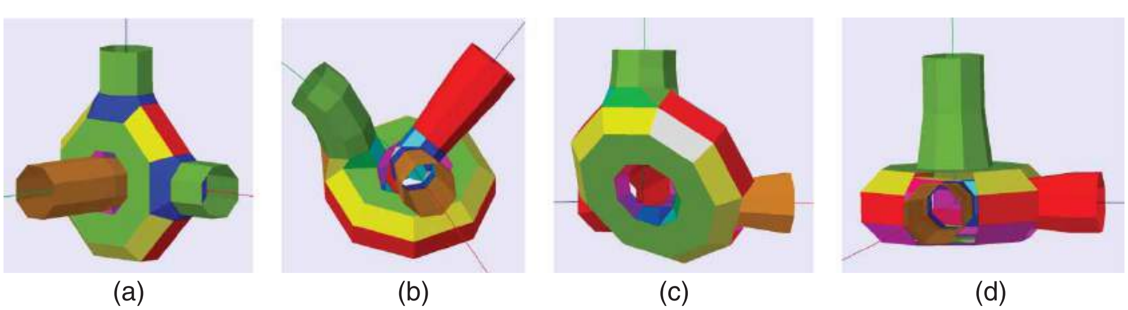


Figure 6. Core geometries of the individual KBM modules depicted in Figure 3.

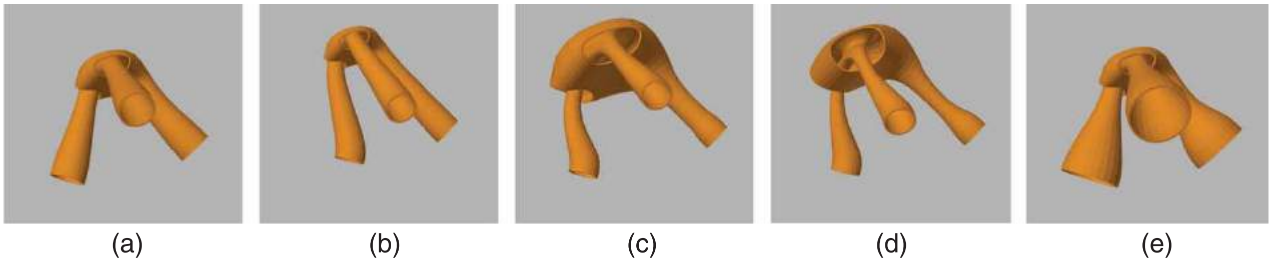


Figure 7. Changing an individual KBM-module: (a) default; (b) KBM displacement; (c) adjusting size and tilt of the toroidal body; (d) changing the inner arm-radius; (e) matching the arm radius at the joints.

One of the two endpoints of such a tubular arm can readily be parameterized so that it will lie at the mid-edge location of the underlying (semi-)regular polyhedral frame, where it connects smoothly to some other KBM-arm with the proper arm-diameter and tangent direction. The other end of this progressive sweep will end up near one of the open facets in the toroidal body or somewhere in its inside, where it may form a Y-junction with another KBM-arm. In either case, at this location some "custom-made" connection geometry has to be defined that guarantees a proper 2-manifold connection between two procedurally generated meshes. The coordinates of the vertices associated with the border of the opening in the toroidal body, generated by the elimination of some of its facets, need to be identified and subjected to all the transformations that the toroidal body has been subjected to in a particular KBM module. Similarly, the vertices of the last octagonal cross-section at the inner end of the progressive sweep used for the arm have to be located, and these two sets of vertices must be connected with a set of triangular and quadrilateral facets that overall have the topology of a cylinder. At this point it is up to the designer, who is adjusting the various free parameters, to make sure that the resulting geometry remains "well behaved" and does not produce any extreme acute dihedral angles or result in unwanted geometrical intersections.

Overall, this parametrization yields a very flexible module, where the dimensions and the tilt of the toroid can be adjusted within reasonable bounds, and where the connected arm can bend and twist so as to terminate in the standardized position given by the edge-midpoint of the polyhedral frame underlying the overall sculpture. Figure 7 shows the effects of varying various parameters: For instance, the two defining radii of the toroidal body may be adjusted individually (Fig. 7b,c); then these torus shells can be re-positioned and tilted (Fig. 5b,d). In addition, the diameters with which the arms emerge from the junction geometry can be adjusted in response to these changes in the toroidal shell (Fig. 7d). The end diameter and directions of these arms are controlled by some global parameters defined for the whole sculpture (Fig. 5e,7e).

4. Tube connections

The design of the structure shown in Figures 1a, 2b and 4a did not present much difficulty. This sculpture is composed of two identical components with arms that end in three mutually parallel tube segments passing through the corners of an equilateral triangle. The curved, connecting tube segments between the two KBM modules are long enough to allow for a very gradual transition from the thin tube emerging from the center of one toroidal ring into the larger tube grafted into the outer surface of another torus. The two modules could readily be connected with three simple cylindrical connector pieces that were inserted by about half an inch into the open tube segments.

It would be nice to have a built-in, "gender-neutral" (no distinction between "male" and "female" ends) connector at the ends of all tubular arm stubs, so that one does not have to bother with inserting extra connector pieces. A natural way to achieve this, given the octagonal cross section of the progressive sweep that form the arms of all the KBM modules, is to attach a protruding prong to every other one of the octagon sides, which then slides into the gap between two such prongs on the matching arm stub forming a junction (Fig. 8a). One disadvantage of such an approach is that it removes the freedom in the azimuthal rotation at which two arms can be joined – which is useful in constructions such as the one shown in Figure 10a. The 4-prong arm stubs can only be joined in four discrete orientations. I chose to preserve the rotational freedom of the tube connections by giving the arm stubs a near-circular symmetry, which can be approximated well enough with two or three levels of Catmull-Clark subdivision of the octagonal prism geometry.

When I started the design of various KBM junction modules for a cube-based assembly (Fig. 3a–d), I did not yet have an understanding of what geometry would make

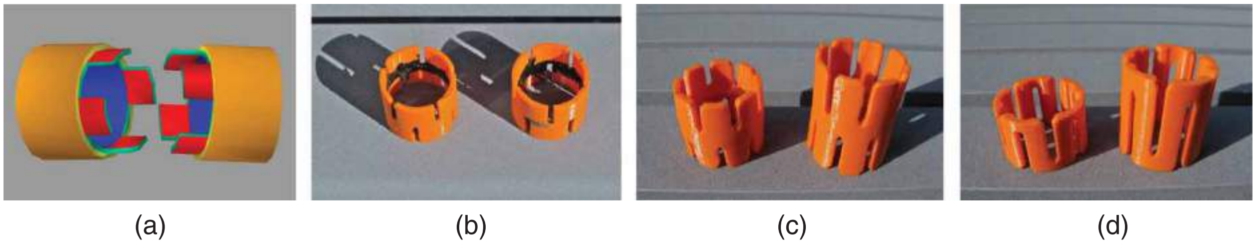


Figure 8. (a) A 4-prong "gender-neutral" connector geometry. (b–d) Various connector types.

the most robust connector element. So I designed the KBM modules with squarely cut off tubular arms, into which I would later insert appropriate tube connectors. I did not want to take the chance that, because of an inadequately designed connector geometry, I would later have to redo all the KBM modules with improved connectors. Instead I focused on designing the smoothest, most organic looking surface modules. Most of them involved some transitions from the "standard arm-diameter" at the joints between two modules to somewhat thinner tubes diving into the inner parts of the toroidal geometry that forms the basis for all the KBM modules, or towards some larger tube diameter leading to the outer envelope of those toroids. In some of the modules these transitions happen over a rather short distance – and this caused a problem: It rendered some tube segments rather conical right up to the joint – rather than keeping them nicely cylindrical. In some of the tubular arms the diameter increases as a distance from the joint; in others it decreases. Thus a simple cylindrical coupling piece no longer makes a good robust connection, and no single connector type can serve all possible combinations of the various conical tube segments. I ended up experimenting with quite a variety of tube connectors – flaring-out by different amounts, and having different stiffness (Fig. 8b–d).

The flexibility of these connector components, built on a FDM (Fused-Deposition Modeling) machine, is limited. Their strength of a particular feature depends on build-orientation. The ideal build orientation for these cylindrical connector parts is with a vertical cylinder axis; this minimizes the amount of support material needed. However, the tensile strength between layers in the z-direction is less than in the other two directions. Thus the type of prongs used in Figures 8b and 8c are prone to break when inserted into a conical arm stub that is too tight. The connector shown in Figure 8d uses a Meander pattern in which the vertical runs have been strengthened with thickened struts. This transforms the bending moment applied to individual prongs into a torsional moment in the horizontal bands at the top and bottom of the cylindrical geometry. However, this type of connector still does not hold well in all combinations of conical arm stubs.

In hindsight, I am glad I did not try to build any of these connector geometries directly into the arms of the KBM modules. – Indeed, I would have had to redo quite a few of them! It is now clear to me, that a better approach is to first design a robust connector system based on truly cylindrical junctions, and then design each KBM module to fit smoothly into three of those connector geometries, properly placed at the mid-points of the edges of the overall chosen polyhedron geometry, as illustrated in Figure 5e. When designing a next set of modular parts, e.g., some new junction modules that would fit an octahedral or dodecahedral edge framework, I will definitely take this preferable approach. I will keep the arm diameters as constant as possible and then adjust the toroidal body, as well as any internal junction geometry, to accommodate the new tubular arms of more uniform thickness. (The impact on the overall aesthetic quality of the sculpture will have to be investigated!)

5. First results: modular sculptures based on cubic graphs

Eight KBM modules selected from the designs shown in Figures 3 and 6 are used to make a sculpture based on a cube frame; I chose to use two instances of each of the four types of modules and placed them at opposite corners of the cube frame. Module 6(b) can readily serve as a stand for this sculpture, balancing it around one of its space diagonals (Fig. 1b). Of course, there are many other possibilities to place the various modules at different corners and to rotate them in place through three possible orientations. In most cases, the resulting surface will be single-sided ($\sigma = 1$) (Fig. 9a); but in a few more symmetrical arrangements, it will remain a two-sided 2-manifold of genus 5 (Fig. 4b).

This first set of branching components can also be used to construct different tubular graph structures, if they are connected via additional short, curved connectors. Three cube-corner components can readily be joined into a 3-ring, by placing between them curved

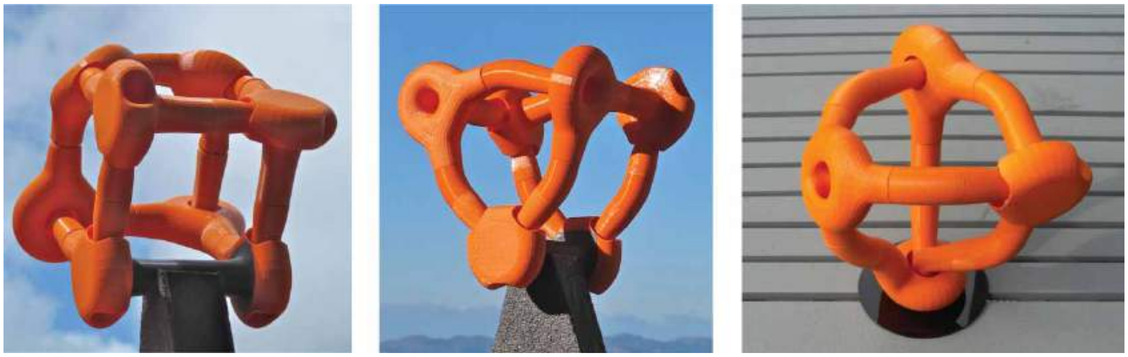

(a) (b) (c)

Figure 9. Sculptures based on cubic graphs: (a) another single-sided cube frame assembly (genus 10), (b) 3-sided prism structure ($\sigma = 1$, genus 8), (c) tetrahedral frame ($\sigma = 1$, genus 6).

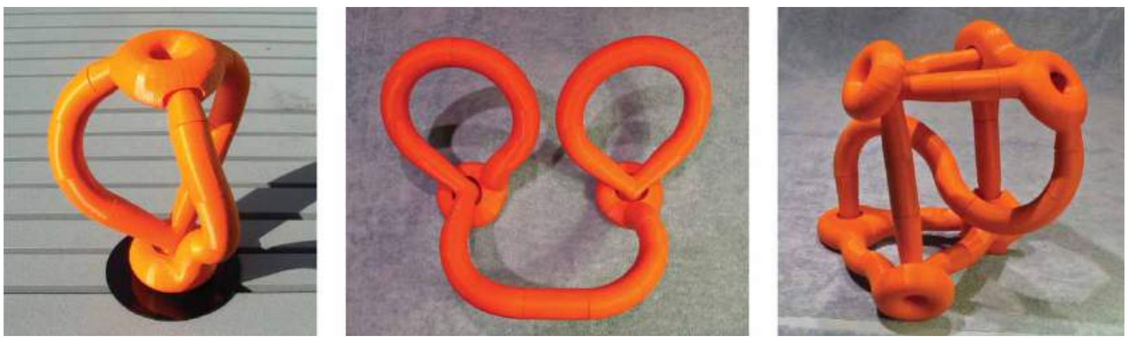

(a) (b) (c)

Figure 10. Less regular structures built from cube-corner modules: (a) 3-arm hosohedron ($\sigma = 1$, $g = 4$), (b) two connected loops ($\sigma = 1$, $g = 4$), (c) distorted 3-sided prism ($\sigma = 1$, $g = 8$).

connectors bending through 30°. Two such rings can then be joined directly to one another to yield a 3-sided prism structure (Fig. 9b). This yields a Super-Bottle of genus $8/\sigma$, where σ is the sidedness of the surface.

The same cube-corner modules can also be used to make a tetrahedral frame out of four of these components. Since the angles between the arms of the cube-corner component are more obtuse than the 60° angles required in the tetrahedron, we need to introduce curved connector pieces that bend through 38.96° to form the curved edges of a tetrahedral frame (Fig. 9c). This yields a Super-Bottle of genus $6/\sigma$.

By making use of the various curved connector pieces introduced so far, some other, less regular tubular sculptures can also be assembled. For example, one could reconstruct the Super-Bottle shown in Figure 1a, by introducing arched connector pieces that bend through 109.48°. However, trying to keep as much modularity with as few different components as possible, I instead reused some of the smaller curved connectors already fabricated to realize that particular structure. Three connectors bending through 38.96° can yield a slightly non-planar connection, with which I can construct a 3-arm hosohedron of genus $4/\sigma$ (Fig. 10a). Alternatively, four connectors bending through 30° could achieve the same result.

Figure 10b shows another assembly that uses only two KBM modules. Rather than joining their arms in three pairs, they are connected via only one arm, and the remaining arms form two local rings. By forming these two loops consistently between either two "thin" arms or two "thick" arms (unlike what is shown in Fig. 10b), the surface could be kept two-sided and orientable, and then would have a genus of only $g = 2$.

Figure 10c depicts a warped 3-sided prism topology; one of the two triangles has been flipped over, and one of the three prism edges now is forced into a contorted "S"-shape.

6. An extension: introducing a 4-way-branch component

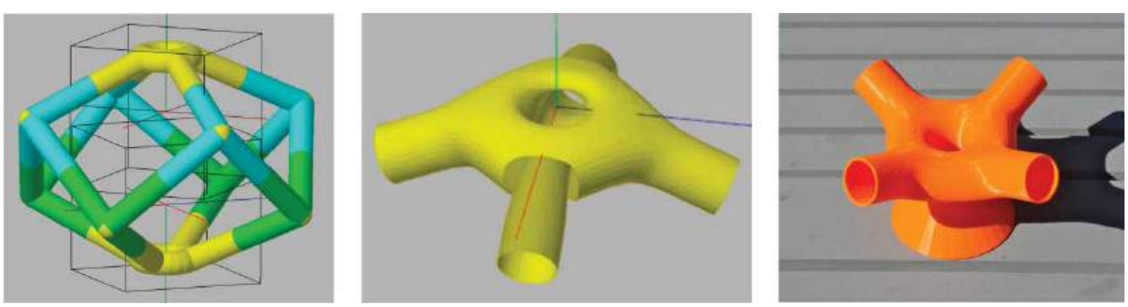
The curved connectors offer no extra design challenges; actually they isolate the problems resulting from any conical arm shapes. New co-design issues arise when a new type of branching element is introduced that represents a node with a different valence.

To build an octahedral frame, a KBM module is needed that has four arms. The appropriate angle between two adjacent arms is 60°. To build the complete configuration, six such modules would have to be fabricated. Instead, to take the next step in my co-design experiments, I chose a slightly different shape that would give me a tangible result after the fabrication of only two new parts, and which also allowed me to evaluate the direct interaction of the new parts with the original cube corner parts without any intermediate curved connector pieces. The resulting tubular structure follows the edges of a polyhedron that could be called a "4-sided anti-pyramid" (Fig. 11a). Removing two opposite corner modules from a complete cube frame (Fig. 1b) leaves a ring of six modules alternatingly tilted by ±54.74° against the 3-fold symmetry axis. By changing this tilt to: arccos(tan 22.5°) = 65.53°, a 4-fold symmetrical ring can be constructed, composed of eight cube corners (Fig. 11a). The four arm stubs pointing towards the same pole can now be joined by a custom-designed 4-way-branch module, in which the four arms bend out of the dominant plane of the torus by 24.47° (Fig. 11b).

In designing this new 4-arm KBM module, some of the design effort expended in the creation of the cube corner module can be re-used. Staying with the general style of such KBM modules, it is natural to connect two of the arms to the outer periphery of a toroidal body, and to feed the other two arms through two individual openings to the inside of the torus, where they join up and merge into the central hole of the torus. The manner in which the adjustable arm-stubs connect to the torus geometry is practically the same as in the module shown in Figures 3c and 6c. The local mesh connectivity is the same – just the numerical values of the vertices change somewhat to accommodate the new angles. In this design I made sure to keep the arms nicely cylindrical.

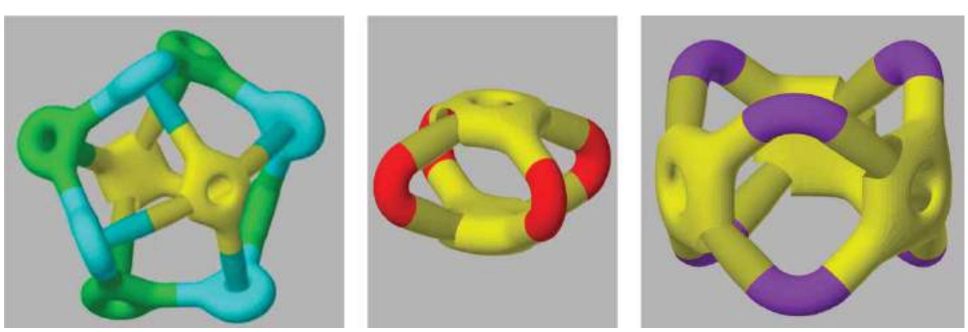
Now that the new 4-way-branch module is available, it can be used to generate several other tubular sculptures that follow different polyhedral frames. Two of the new modules can be connected into a 4-arm hosohedron with curved connectors that bend through 131.06°, i.e., twice the complement of 24.47° (Fig. 12b). Three or more of these 4-way-branch modules can be joined into doubly-linked rings. To form a 3-component ring, connectors must bend through 94.2° (Fig. 12c). A 4-part ring requires connectors that bend through 83.2°; and in a 5-module ring the connectors bend through 80.3° (Fig. 13a).

To provide an "organically" integrated stand for some of these sculptures, I have created a modified version of



(a) (b) (c)

Figure 11. Introducing a 4-branch KBM junction: (a) calculating the branching angle for a D_4-symmetric polyhedron; (b) resulting 4-arm KBM module; (c) modified KBM module that can serve as a stand.



(a) (b) (c)

Figure 12. Tubular graph structures enabled by the 4-branch KBM junction element: (a) genus $14/\sigma$, (b) genus $6/\sigma$, (c) genus $8/\sigma$.

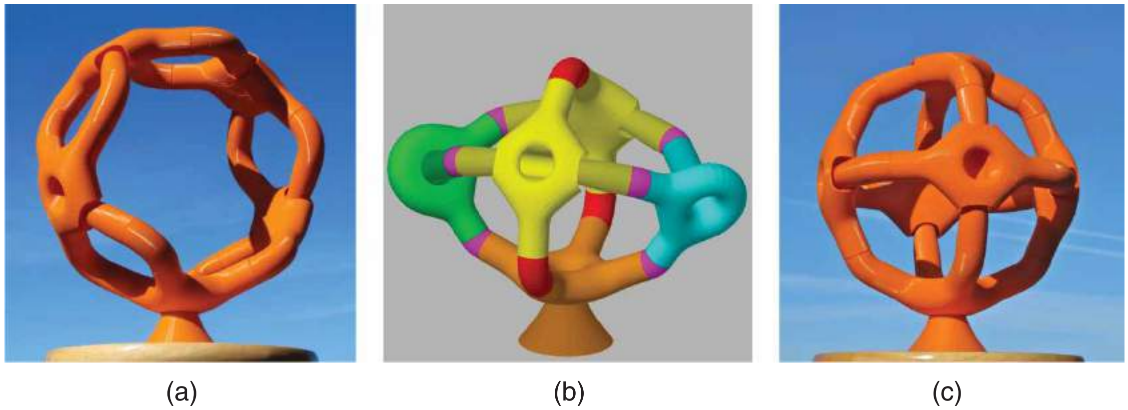


Figure 13. More tubular graph structures enabled by the 4-branch KBM junction element: (a) genus $12/\sigma$, (b) genus $10/\sigma$, (c) genus $14/\sigma$

the new 4-arm branching module: All four arms connect to the outer periphery of the toroidal body, and a cone shape is fused into the inner hole of the torus (Fig. 11c). This cone then allows some of the sculptures to be balanced on this valence-4 junction module (Fig. 13a-c).

Another way to combine the old and the new branching modules joins three of the new valence-4 modules into a singly-linked ring with connectors that bend through 71.06°; the remaining open arm stubs can then be connected to two regular cube-corner modules (shown in green and blue) with connectors that bend through 30.24°. Once again this sculpture could be mounted on the special extended 4-arm module (Fig. 13b). Finally, an octahedral frame can still be obtained by connecting six of these valence-4 modules with small connector pieces that bend through 41.06° (Fig. 13c); this results in a 2-manifold of genus $14/\sigma$.

7. A system of coupling pieces

The above bent coupling pieces between the various KBM parts were designed pretty much on demand and in an ad-hoc manner. The angle through which a coupler has to bend is given by the geometry of the underlying (regular) polyhedron; but the radius at which it does this is an open parameter. For the sculptures shown in Figure 13, the bending radii were chosen to be relatively small, in order to produce somewhat compact configurations. In retrospect, we might want a more modular and more logical system of coupling pieces with less arbitrarily defined bending radii.

In Figures 9b and 10c two planar loops are formed by three pairs of KBM arms and three coupler pieces. This loop could be seen as a single circular ring of constant radius onto which the KBM modules fit tangentially with a pair of their arms. The radius of this circle would then be equal to the length of a standard KBM arm measured from the triple-junction point of their axes. Four corner pieces emulate that circle without the need for additional coupling pieces. With only three corner pieces in the loop, two additional couplers bending through 45° would complete the ring, or three pieces of 30° could complete the loop with full symmetry.

In other sculptures, the corner pieces and couplers are in a more “3-dimensional” arrangement. In this case the KMB module could be understood as fitting with all three arms tangentially onto a sphere. For the 8-component sculpture this sphere touches the midpoints of the cube frame, and its radius is thus $\sqrt{2}$ times the length of a standard KBM arm. This sphere could be used as the underlying basis for the radius of all coupling pieces participating in non-planar loops. For the 3-arm hosohedron (Fig. 10a) we would need three coupling pieces bending through 109.48°, and for the tetrahedron (Fig. 9c), six couplers with angles of 38.96° are required.

It is rather inconvenient that we need all these curved connectors, which all bend through somewhat different angles. A purely functional, topological solution could be obtained by using some short pieces of flexible hoses or some tubular ball-and-socket-joint arrangement that can bend through a range of angles. This would be good enough to establish the desired connectivity for topological studies. However, in order to compose aesthetically pleasing sculptures, a better, more “organic” match between the junction modules and the connector pieces is desirable.

At the next higher level of a co-design of such a system of modular parts, on must ask what is the best collection of components that together form a complete, versatile “LEGO® -like” building block set? Once I had fabricated eight cube corner modules, as well as the six 4-arm modules needed for the octahedral frame (Fig. 13c), I naturally wanted to combine them into a tubular frame based on the rhombic dodecahedron. It turned out that, based on the existing components, this requires 24 coupler pieces bending through 4.99°; all 24 edges would then be slightly "concave" and bend inwards towards the center of the polyhedron (Fig. 14). Thus, in hindsight, it would have been advantageous, to slope the arms of the 4-arm module by only 19.48°, so that these 14 modules could be joined directly. Of course, this then requires coupling pieces in the assembly of Figure 12a – but there are only eight connections in that one!

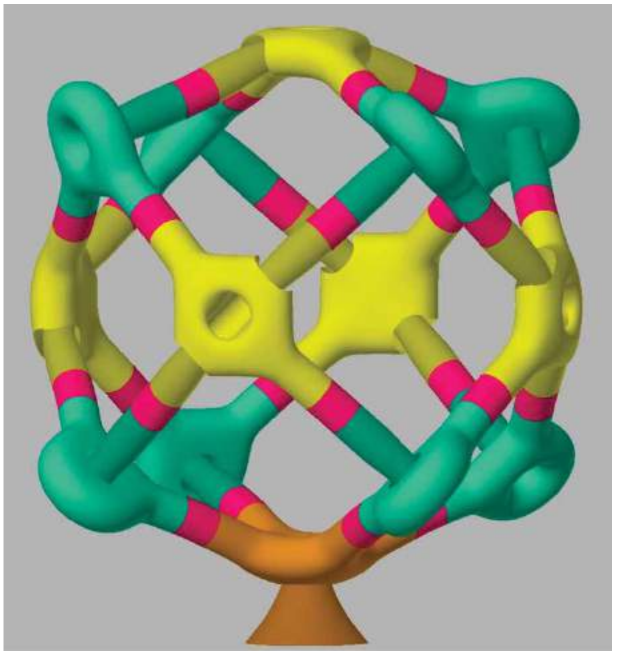


Figure 14. Rhombic dodecahedron (genus 22/σ).

8. Summary and conclusions

Parametric design is powerful. It allows quick modification and fine-tuning of an assembly, while looking at the final result. As stated earlier, robust parametrization is difficult even for a single object, but becomes particularly hard, when a system is composed of several components that are designed to be combined in many different ways. This paper outlines this iterative design process for a small set of free-form modules that enable the composition of single-sided or double-sided 2-manifolds of higher genus.

The first set of 3-way junction modules were optimized with an overall cube-frame structure in mind. If one now wants to extend the building block set to allow a structure following a dodecahedral frame, the angles between the three tubular arms would be too small. New modules might have to be introduced in which the arms spread by 108° between them. This may require more than just a parametric change; the polyhedral meshes defining the KBM modules may have to be changed slightly, e.g. a different set of faces on the toroidal shell would have to be removed to allow entry of one or two thin tubular arms at an optimal angle. In practice, appropriate parameterization for easy interactive optimization can only go so far; it can only cover a limited range of adjustments.

On the other hand, during the design of the 4-arm KBM module, we found that much of the development effort from the original cube corner could be carried forward and re-used. The resulting parameterization of the new module then could be made quite similar and highly compatible with the older one. Thus, the overall system design and optimization could have readily been carried out with both of these components in play.

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