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Fairness metric of plane curves defined with similarity geometry invariants

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ABSTRACT
A curve is considered fair if it consists of continuous and few monotonic curvature segments. Polynomial curves such as Bézier and B-spline curves have complex curvature function, hence the curvature profile may oscillate easily with a little tweak of control points. Thus, bending energy and shear deformation energy are common fairness metrics used to produce curves with monotonic curvature profiles. The fairness metrics are used not just to evaluate the quality of curves, but it also aids in reaching to the final design. In this paper, we propose two types of fairness metric functionals to fair plane curves defined by the similarity geometry invariants, i.e. similarity curvature and its reciprocal to extend a variety of aesthetic fairing metrics. We illustrate numerical examples to show how log-aesthetic curves change depending on α and G^1 constraints. We extend LAC by modifying the integrand of the functionals and obtain quasi aesthetic curves. We also propose σ-curve to introduce symmetry concept for the log-aesthetic curve.

KEYWORDS
Fairness metric; Plane curve; Similarity geometry; Invariants; Log-aesthetic curve

1. Introduction

A curve is considered fair if it consists of continuous and few monotonic curvature segments [2]. Polynomial curves such as Bézier and B-spline curves have complex curvature function, hence the curvature profile may oscillate easily with a little tweak of control points. Thus, bending energy and shear deformation energy are common fairness metrics used to produce curves with monotonic curvature profiles. The fairness metrics are used not just to evaluate the quality of curves, but it also aids in reaching to the final design.

Curve synthesis is a process of generating curves with a well-defined Cesáro equation, which describes the curvature κ of a curve as a function of its arc length s. Log-aesthetic curves (LAC in short) [5] are generated with a Cesáro equation derived by letting the Logarithmic curvature graph (LCG) as a linear function with the gradient as α. This curve has gained its momentum in design environment and now is it used for automobile [7] and architecture [13] design. The family of LACs includes logarithmic (equiangular) curves ($\alpha = 1$), clothoid curves ($\alpha = -1$), circle involutes ($\alpha = 2$) and Nielsen's spiral ($\alpha = 0$). It is possible to generate and deform LACs in real time regardless of its integral forms using their unit tangent vectors as integrands when $\alpha \neq 1, 2$.

Recently, Sato and Shimizu [9] expressed LACs by a simple equation in similarity geometry where the direction angle θ of a given curve is invariant. For a given curve $C(\theta) = (x(\theta), y(\theta))$, the similarity curvature $S(\theta) \equiv -\rho_\theta/\rho$ is also invariant where ρ is radius of curvature and $\rho_\theta = d\rho/d\theta$. The slope of the LCG of a LAC can be expressed by

$$\alpha = \frac{S_\theta}{S^2} + 1 \tag{1.1}$$

Thus the similarity curvature of LAC satisfies the following Riccati (Bernoulli) equation:

$$S_\theta = (\alpha - 1)S^2 \tag{1.2}$$

The above equation can be solved easily to obtain the similarity curvature of LAC as follows:

$$S(\theta) = \frac{-1}{(\alpha - 1)\theta + c} \tag{1.3}$$

where c is an integral constant.

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In this paper, we propose two types of fairness metric functionals to fair plane curves defined by the similarity geometry invariants, i.e. similarity curvature and its reciprocal to extend a variety of aesthetic fairing metrics. Section 4 also illustrates numerical examples to show how LACs changes depending on α and G^1 constraints. In section 5, we introduce a modified functional and symmetry concept for LAC.

2. Similarity geometry

We may deduce to figures similar each other when these figures possess the same shape even if their sizes are different. In similarity geometry if two objects are similar, then we deduce that both are equivalent. In Euclidean geometry, circles with different radii are considered different entities, but in similarity geometry circles with different radii are regarded as the same.

In this section, we derive similarity Frenet frame to introduce the definition of similarity curvature and show its role in similarity geometry [3]. Since we know that the arc length s may vary, thus the representation of plane curves is parameterized by direction angle θ which is invariant by scaling. First, let a plane curve be given as a function of its arc length by

$$C(s) = (x(s), y(s)) \tag{2.1}$$

and its Frenet frame $F(s) = (T(s), N(s))$. We assume the curve is not a straight line and the direction angle θ is defined by

$$\theta = \int_0^s \kappa(s)ds \tag{2.2}$$

where κ is curvature. Next, let tangent vector $T^{Sim}(\theta)$ as follows to define the Frenet frame in similarity geometry,

$$T^{Sim}(\theta) \equiv \frac{dC}{d\theta}(\theta) \tag{2.3}$$

Thus, we may simplify as

$$T^{Sim}(\theta) = \frac{dC}{ds}\frac{ds}{d\theta} = \frac{1}{\kappa(s)}T(s) \tag{2.4}$$

where $T(s)$ is the first derivative of $C(s)$ with respect to s and it is a unit tangent vector of the curve. Let $N^{Sim}(\theta)$ be

$$N^{Sim}(\theta) = \frac{1}{\kappa(s)}N(s) \tag{2.5}$$

Since $\det(T^{Sim}, N^{Sim}) = 1/\kappa^2$, hence $F^{Sim}(\theta) = (T^{Sim}(\theta), N^{Sim}(\theta))$ has a value in

$$CO^+(2) = \{X \in CO(2) | \det X > 0\} \tag{2.6}$$

where $CO^+(2)$ is a set of 2×2 real matrix A such that $AA^T = cE$ for an arbitrary constant c. Here A^T denotes a transpose of matrix A and E does a unit matrix. The derivatives of $T^{Sim}(\theta)$ and $N^{Sim}(\theta)$ are given by

$$\frac{d}{d\theta}T^{Sim}(\theta) = -\frac{\kappa_s(s)}{\kappa(s)^2}T^{Sim}(\theta) + N^{Sim}(\theta) \tag{2.7}$$

$$\frac{d}{d\theta}N^{Sim}(\theta) = -\frac{\kappa_s(s)}{\kappa(s)^2}N^{Sim}(\theta) - T^{Sim}(\theta) \tag{2.8}$$

From equations (2.7) and (2.8), we define

$$S(\theta) = \frac{\kappa_s(s)}{\kappa(s)^2} \tag{2.9}$$

Equation (2.9) is an invariant in similarity geometry and it is denoted as similarity curvature. Therefore, $F^{Sim}(\theta)$ satisfies the following differential equation:

$$\frac{d}{d\theta}F^{Sim}(\theta) = F^{Sim}(\theta)\begin{pmatrix} -S(\theta) & -1 \\ 1 & -S(\theta) \end{pmatrix} \tag{2.10}$$

The above equation is called the formula of Frenet frame in similarity geometry.

3. Similarity geometry invariants

As stated in the previous section, by regarding direction angle θ as a function of arc length s, the similarity curvature $S(\theta(s))$ is defined by

$$S(\theta(s)) = \frac{1}{\kappa(s)^2}\frac{d\kappa}{ds} = -\frac{d\rho}{ds} \tag{3.1}$$

Similarity radius of curvature $V(\theta(s))$ is defined as a reciprocal of similarity curvature $S(\theta(s))$ and it is derived as follows

$$V(\theta(s)) = \frac{1}{S(\theta(s))} = \frac{\kappa(s)^2}{\frac{d\kappa}{ds}} = -\frac{1}{\frac{d\rho}{ds}} \tag{3.2}$$

In this paper, two types of functionals are proposed to fair a plane curve $C(t)$ whose domain is $[a, b]$. The first type is given by

$$F_{sc}(C(t)) = \int_a^b S(\theta(t))^2\frac{d\theta}{dt}dt \tag{3.3}$$

and the second type is

$$F_{sroc}(C(t)) = \int_a^b V(\theta(t))^2\frac{d\theta}{dt}dt \tag{3.4}$$

Eqns. (3.3) and (3.4) are now rewritten as follows:

$$F_{sc}(C(t)) = \int_0^l \frac{1}{\kappa(s)^4}\left(\frac{d\kappa}{ds}\right)^2\kappa(s)ds = \int_0^l \frac{1}{\kappa(s)^3}\left(\frac{d\kappa}{ds}\right)^2 ds$$
$$= \int_0^l \frac{1}{\rho(s)}\left(\frac{d\rho}{ds}\right)^2 ds \tag{3.5}$$

where l is a total length of curve $C(t)$. Similarly, the second type is rewritten as follows:

$$F_{sroc}(C(t)) = \int_0^l \frac{\kappa(s)^5}{\left(\frac{d\kappa}{ds}\right)^2} ds = \int_0^l \frac{1}{\rho(s)\left(\frac{d\rho}{ds}\right)^2} ds \quad (3.6)$$

Consider two traditional functionals commonly used for fairing plane curves:

$$\int_0^l \kappa^2(s) ds \quad (3.7)$$

called bending energy and

$$\int_0^l \left(\frac{d\kappa}{ds}\right)^2 ds \quad (3.8)$$

called shear deformation energy. These functionals are clearly different from F_{sc} or F_{sroc}.

4. Euler-Lagrange equation [1]

4.1. Similarity curvature

From Eqn. (3.5):

$$F_{sc}(C(t)) = \int_0^l f_{sc}(s) ds = \int_0^l \frac{1}{\kappa(s)^3}\left(\frac{d\kappa}{ds}\right)^2 ds \quad (4.1)$$

Under suitable boundary conditions its Euler-Lagrange equation in terms of κ is

$$\frac{\partial f_{sc}}{\partial \kappa} - \frac{d}{ds}\frac{\partial f_{sc}}{\partial \dot{\kappa}} = -3\frac{\dot{\kappa}^2}{\kappa^4} - 2\frac{d}{ds}\frac{\dot{\kappa}}{\kappa^3} = \frac{3}{\kappa^4}\left(\dot{\kappa}^2 - \frac{2}{3}\kappa\ddot{\kappa}\right) = 0 \quad (4.2)$$

where $\dot{g} = dg/ds$ and $\ddot{g} = d^2g/ds^2$ for function g of s. It is known that LACs satisfy the following equation [5]:

$$\kappa^{-\alpha} = cs + d \quad (4.3)$$

where c and d are constants. We obtain Eqn. (4.4) after differentiating both sides of the above equations twice:

$$-\alpha(-\alpha-1)\kappa^{-\alpha-2}\dot{\kappa}^2 - \alpha\kappa^{-\alpha-1}\ddot{\kappa} = 0 \quad (4.4)$$

If $\alpha \neq 0$ and $-\alpha - 1 \neq 0$ ($\alpha \neq -1$), then

$$\dot{\kappa}^2 - \frac{1}{\alpha+1}\kappa\ddot{\kappa} = 0 \quad (4.5)$$

By comparing Eqns. (4.2) and (4.5), the curve which minimizes Eqn. (4.1) is a log-aesthetic curve whose α is equal to $1/2$. This fact demonstrates that LAC can be expressed by a simple similarity curvature, which has a natural property and plays an important role in similarity geometry.

On the other hand, from

$$F_{sc}(C(t)) = \int_0^l \frac{1}{\rho(s)}\left(\frac{d\rho}{ds}\right)^2 ds \quad (4.6)$$

its Euler-Lagrange equation in terms of ρ is

$$\frac{\partial f_{sc}}{\partial \rho} - \frac{d}{ds}\frac{\partial f_{sc}}{\partial \dot{\rho}} = -\frac{\dot{\rho}^2}{\rho^2} - 2\frac{d}{ds}\frac{\dot{\rho}}{\rho} = \frac{1}{\rho^2}(\dot{\rho}^2 - 2\rho\ddot{\rho}) = 0 \quad (4.7)$$

Similarly, Eqns. (4.3) and (4.4) are rewritten with ρ

$$\rho^{\alpha} = cs + d \quad (4.8)$$

and

$$\alpha(\alpha-1)\rho^{\alpha-2}\dot{\rho}^2 + \alpha\rho^{\alpha-1}\ddot{\rho} = 0 \quad (4.9)$$

are satisfied by LACs. If $\alpha \neq 0$ and $\alpha - 1 \neq 0$ ($\alpha \neq 1$), then

$$\dot{\rho}^2 + \frac{1}{\alpha-1}\rho\ddot{\rho} = 0 \quad (4.10)$$

By comparing Eqns. (4.7) and (4.10), the curve which minimizes Eqn. (4.6) is a log-aesthetic curve whose α is equal to $1/2$. This result is consistent to that of the curvature formulation shown above.

4.2. Similarity radius of curvature

From Eqn. (3.5)

$$F_{sroc}(C(t)) = \int_0^l f_{sroc}(s) ds = \int_0^l \frac{\kappa(s)^5}{\left(\frac{d\kappa}{ds}\right)^2} ds \quad (4.11)$$

its Euler-Lagrange equation in terms of κ is

$$\frac{\partial f_{sroc}}{\partial \kappa} - \frac{d}{ds}\frac{\partial f_{sroc}}{\partial \dot{\kappa}} = \frac{\kappa^4}{\dot{\kappa}^2} + 2\frac{d}{ds}\frac{\kappa^5}{\dot{\kappa}^3}$$
$$= 15\frac{\kappa^4}{\dot{\kappa}^4}\left(\dot{\kappa}^2 - \frac{2}{5}\kappa\ddot{\kappa}\right) = 0 \quad (4.12)$$

By comparing Eqns. (4.5) and (4.12), the curve which minimizes Eqn. (4.11) is LACs whose α is equal to $3/2$. From

$$F_{sroc}(C(t)) = \int_0^l f_{sroc}(s) ds = \int_0^l \frac{1}{\rho(s)\left(\frac{d\rho}{ds}\right)^2} ds \quad (4.13)$$

its Euler-Lagrange equation in terms of κ is

$$\frac{\partial f_{sroc}}{\partial \rho} - \frac{d}{ds}\frac{\partial f_{sroc}}{\partial \rho\kappa} = -\frac{3}{\rho^2\dot{\rho}^4}(\dot{\rho}^2 + 2\rho\ddot{\rho}) = 0 \quad (4.14)$$

By comparing Eqns. (4.10) and (4.14), the curve which minimizes Eqn. (4.13) is LACs whose α is equal to $3/2$. Again, this result is consistent to that of the curvature formulation shown above.

4.3. General n

So far, we have shown that to minimize functionals defined by similarity invariants, we obtain log-aesthetic curves with specific α values. This fact strongly suggests us that we might obtain LACs with other α values by minimizing variant functionals using similarity invariants. In this section, we define a functional using a general real number $n(n \neq 0)$ as follows:

$$F_{gen}(C(t)) = \int_0^l f_{gen}(s)ds = \int_0^l S(\theta(s))^n \kappa(s)ds$$
$$= \int_0^l \left(\frac{1}{\kappa(s)^2}\frac{d\kappa}{ds}\right)^n \kappa(s)ds = \int_0^l \frac{1}{\kappa(s)^{2n-1}}\left(\frac{d\kappa}{ds}\right)^n ds \tag{4.15}$$

Its Euler-Lagrange equation in terms of κ is

$$\frac{\partial f_{gen}}{\partial \kappa} - \frac{d}{ds}\frac{\partial f_{gen}}{\partial \dot{\kappa}} = (2n-1)(n-1)\frac{\kappa^{n-2}}{\dot{\kappa}^{2n}}$$
$$\times \left(\dot{\kappa}^2 - \frac{n}{2n-1}\kappa\ddot{\kappa}\right) = 0 \tag{4.16}$$

By comparing Eqns. (4.5) and (4.16), the curve which minimizes Eqn. (4.15) is

$$\frac{1}{\alpha+1} = \frac{n}{2n-1} \tag{4.17}$$

The above equation yields

$$\alpha = 1 - \frac{1}{n} \tag{4.18}$$

Note that the above equation tells us that when $n = 2$ (corresponding similarity curvature), $\alpha = 1/2$ and when $n = -2$ (corresponding similarity radius of curvature), $\alpha = 3/2$, that are consistent to the discussion in the previous sections.

If $\alpha < 0$, the LAC can have an inflection point. The condition for $\alpha < 0$ is as follows:

$$1 - \frac{1}{n} < 0 \tag{4.19}$$

If $n > 0$, then $n < 1$. Hence we have a solution if $0 < n < 1$. If $n < 0$, then $n > 1$. In this case, there is no solution. To summarize, if $0 < n < 1$, α is negative.

Suzuki et al. [11] proposed a functional called K_{LAC} based on the shortest path in the aesthetic space [6] which is minimized by a LAC as follows:

$$K_{LAC} = \int_0^l (\sigma_s)^2 ds \tag{4.20}$$

where $\sigma = \rho^\alpha$ and $\sigma_s = d\sigma/ds = \alpha\rho^{\alpha-1}\rho_s$. The above functional can be rewritten as follows:

$$K_{LAC} = \alpha^2 \int_0^l (\rho^{\alpha-1}\rho_s)^2 ds = \left(1 - \frac{1}{n}\right)^2 \int_0^l \left(\frac{1}{\rho}\rho_s^n\right)^{\frac{2}{n}} ds \tag{4.21}$$

The integrand without power $2/n$ of the rightest expression in the above equation is equal to that of the functional given by Eqn. (4.15) even though it is given by ρ and its derivative ρ_s. Note that although K_{LAC} is scale-variant, that means if we deform a curve by minimizing it, we obtain different curves depending on their sizes, the functional in Eqn. (4.15) is scale-invariant in nature and we obtain the same shaped curves independent from their sizes. Because of constraints, the curve generated by minimizing this functional might not converge to a LAC segment, but we guarantee the scale invariance of the generated curve by use of this functional. On scale invariance of LACs please refer to Suzuki et al [12].

4.4. Numerical examples

Figure 1 shows the comparisons of LAC shapes whose $\alpha = 1/2$ and $3/2$. It consists of three pairs of LACs and the curves in each pair are generated with the same G^1 constraints. When the G^1 constraints vary drastically from

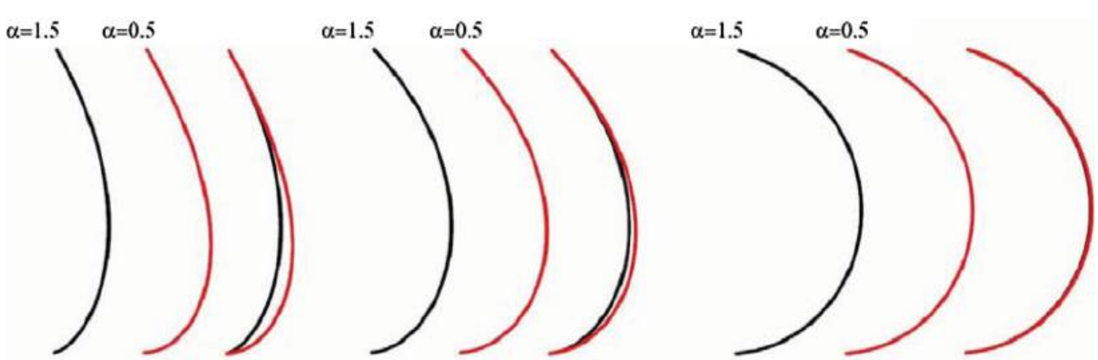


Figure 1. Comparisons of LACs whose $\alpha = 1/2$ and $3/2$.

those for a circular arc, then the LAC shapes become distinctly different. Although the differences of their shapes are somehow restricted, switching α from 1/2 to 3/2 and vice versa provides a subtle deformation of the curve.

5. Extensions of the Log-aesthetic curve

So far we have shown that similarity geometry and $\sigma = \rho^\alpha$ have main roles to define the log-aesthetic curve. In this section, we extend LAC in two ways: The 1st is to modify the integrand of the functional and the 2nd is to define curve in the aesthetic space.

5.1. Modification of the integrand

We modify the integrand of the functional in Eqn. (4.15) as follows:

$$F_{\lambda,a}(C(\theta)) = \int_{\theta_0}^{\theta_1} \frac{1}{2}\left(a^2 S(\theta)^2 + \frac{\gamma}{q^{2a}}\right) d\theta \tag{5.1}$$

where a and γ are constants and $q^2 = T^{Sim}(\theta) \cdot T^{Sim}(\theta)$. Its Euler-Lagrange equation under suitable boundary conditions is given by

$$\frac{dS}{d\theta} = aS^2 + c \tag{5.2}$$

where c is a constant. When $a = -1/2$ and $\lambda = 4c$, the above equation becomes the following Riccati equation

$$\frac{dS}{d\theta} = -\frac{1}{2}S^2 + \frac{\lambda}{4} \tag{5.3}$$

and Shimizu and Sato gave the solutions for it as follows [10]:

$$S(\theta) = -\sqrt{-\frac{\lambda}{2}} \tan\left(\sqrt{-\frac{\lambda}{2}}\frac{\theta}{2} - \theta_0\right) \text{ if } \lambda < 0 \tag{5.4}$$

$$S(\theta) = \sqrt{\frac{\lambda}{2}} \tanh\left(\sqrt{\frac{\lambda}{2}}\frac{\theta}{2} + \theta_0\right) \text{ or} \tag{5.5}$$

$$S(\theta) = \sqrt{\frac{\lambda}{2}} \coth\left(\sqrt{\frac{\lambda}{2}}\frac{\theta}{2} + \theta_0\right) \text{ if } \lambda > 0 \tag{5.6}$$

They called these curves the quasi aesthetic curves [9]. In Eqn. (5.3) if $\lambda = -8$, then from Eqn. (5.4) the similarity curvature of the curve becomes

$$S(\theta) = -2\tan(\theta - \theta_0) \tag{5.7}$$

Assuming $\theta_0 = 0$, the above formula corresponds to the similarity curvature of the catenary curve expressed by

$$y = \frac{a}{2}\left(\exp\left(\frac{x}{a}\right) + \exp\left(-\frac{x}{a}\right)\right) \tag{5.8}$$

where a is a positive constant. Its radius of curvature ρ satisfies the following equation:

$$\rho^{-\frac{1}{2}} = \frac{1}{\sqrt{a}}\cos(\theta) \tag{5.9}$$

for $-\pi/2 < \theta < \pi/2$. If we measure arc length s from $(x, y) = (0, a)$ to the positive x direction, s is given by

$$s = \int_0^x \sqrt{1 + (y')^2}dx = \frac{a}{2}\left(\exp\left(\frac{x}{a}\right) - \exp\left(-\frac{x}{a}\right)\right)$$
$$= a\tan(\theta) \tag{5.10}$$

From Eqns. (5.9) and (5.10),

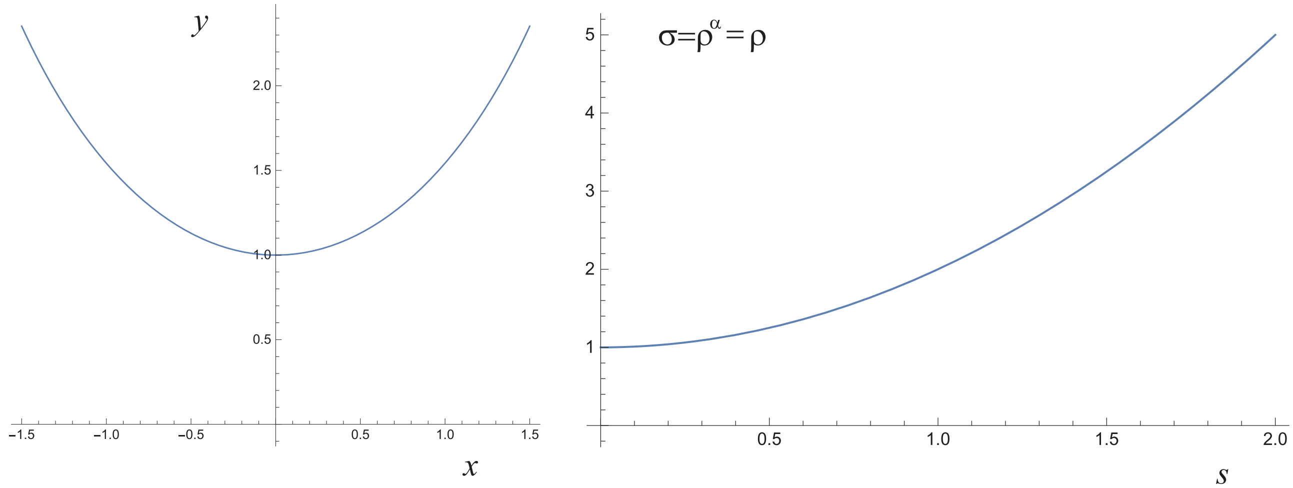
$$\rho = \frac{1}{a}s^2 + a \tag{5.11}$$

Therefore, the radius of curvature ρ of catenary curve can be described as a quadratic function of its arc length s in the $\alpha = 1$-aesthetic space, which we explain in the next section, and we can say that it is one of σ-curves proposed in the next section. Figure 2 depicts a catenary curve with $a = 1$ and the same curve in the $\alpha = 1$-aesthetic space. Note that the catenary curve is symmetric along the y axis and we also introduce symmetry concept in the next section.

5.2. σ-Curves

In this section, we extend LAC in the aesthetic space where the horizontal axis is given by arc length s and the vertical axis is by σ to improve its expressive power. A LAC segment is expressed by a straight-line segment in the aesthetic space as shown in Fig. 3. We call a new curve σ-curve because it is defined using σ values. The main purpose to propose a new curve is to introduce symmetry for curve generation. Symmetry is one of the most important concepts for aesthetic design (for example, refer to Maor and Jost [4]).

Curves are a very basic element for aesthetic design and the monotonicity of curvature of the curve is regarded as a desirable characteristic for its fairness and beauty [8]. However, if we guarantee its curvature monotonicity, we cannot generate a symmetrical curve and we cannot achieve a design with symmetry. Although we can generate two segments which are symmetrical each other and connect them, we lose smoothness between them, i.e. we generally lose G^3 and higher continuity.



(a) a catenary curve (b) the curve in (a) in the aestheic space

Figure 2. A catenary curve ($a = 1$) in x-y plane and in the aesthetic space with $\alpha = 1$.

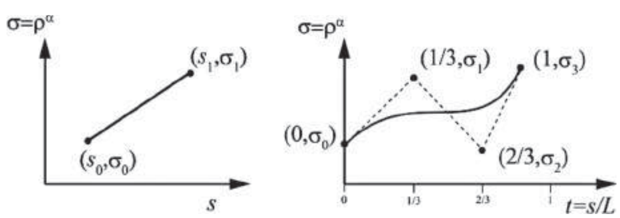


Figure 3. The aesthetic space (left and right graphs correspond to a LAC and a σ-curve, respectively).

Hence to generate a symmetrical curve in a more natural way, we propose σ-curve defined in the aesthetic space. The curve is regarded as a LAC version of the ALP (Arc-Length Parametrization) curve recently proposed by Yoshida and Saito [14]. Note that σ is not a similarity geometry invariant.

5.2.1. Definition of σ-curves

We define σ curve by its Cesàro equation as follows:

$$\sigma = \rho^{\alpha} = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 \quad (5.12)$$

In the above equation, $\sigma = \rho^{\alpha}$ is given by a polynomial function of arc length s.

5.2.2. Bézier, B-spline and NURBS types

If the total length L of a curve is given, Bézier, B-spline and NURBS types of the σ-curve can be defined in a straightforward manner. Bézier type is defined by

$$\rho(s)^{\alpha} = \sum_{i=0}^{n} B_i^n\left(\frac{s}{L}\right)\sigma_i \quad (5.13)$$

where $B_i^n(t)$ is a Bernstein basis function of degree n and $\sigma_i = \rho_i^{\alpha}$. The right graph in Fig. 3 shows σ values of a Bézier σ-curve. B-spline type is

$$\rho(s)^{\alpha} = \sum_{i=0}^{n} N_i^n\left(\frac{s}{L}\right)\sigma_i \quad (5.14)$$

where $N_i^n(t)$ is a B-spline basis function of degree n. NURBS-spline type is

$$\rho(s)^{\alpha} = \frac{\sum_{i=0}^{n} N_i^n\left(\frac{s}{L}\right) w_i\sigma_i}{\sum_{i=0}^{n} N_i^n\left(\frac{s}{L}\right) w_i} \quad (5.15)$$

5.2.3. Generalized σ-curves

We can perform an extension of σ-curve in a similar way from the log-aesthetic curve to the generalized log-aesthetic curve (GLAC). There are two types of the LAC: 1st type is given by $\rho^{\alpha} = c_1 s + c_0$ and 2nd type $\rho = \exp(c_1 S + c_0) = C_0\exp(c_1 S)$ [5]. Note that the 2nd type of the LAC also satisfies Eqn. (1.2) in case of $\alpha = 0$. For the 1st type we can have two types of generalized σ-curve, i.e. ρ-shift and κ-sift:

$$(\rho - e)^{\alpha} = a_n s^n + a_{n-1} s^{n-1} \cdots + a_1 s + a_0 \quad (5.16)$$

$$(\kappa - f)^{-\alpha} = b_n s^n + b_{n-1} s^{n-1} \cdots + b_1 s + b_0 \quad (5.17)$$

where e, f and $a_i, b_i (i = 0, \cdots n)$ are constants. We call n degree of the σ-curve. For the 2nd type we can have

$$\begin{aligned}\rho - e &= \exp(a_n s^n + a_{n-1} s^{n-1} \cdots + a_1 s + a_0)\\ &= A_0\exp(a_n s^n + a_{n-1} s^{n-1} \cdots + a_1 s) \quad (5.18)\end{aligned}$$

$$\begin{aligned}\kappa - f &= \exp\{-(b_n s^n + b_{n-1} s^{n-1} \cdots + b_1 s + b_0)\}\\ &= D_0 \exp(d_n s^n + d_{n-1} s^{n-1} \cdots + d_1 s) \quad (5.19)\end{aligned}$$

where $D_0 = \exp(-b_0)$ and $d_i = -b_i (i = 0, \cdots n)$ are constant.

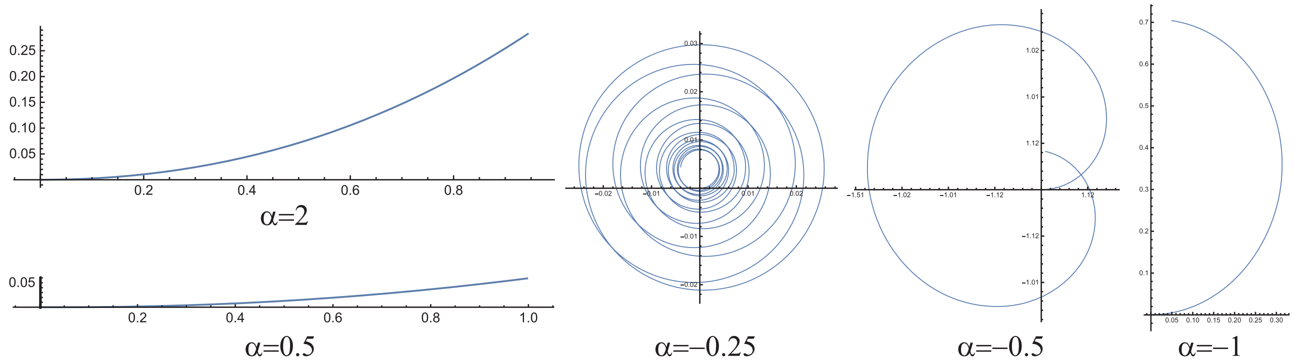


Figure 4. σ-curves with various α values.

5.2.4. Symmetrical curve examples

We will show an example of a symmetrical Bézier σ-curve in this section. To make a Bézier σ-curve of degree n symmetric, the following condition is necessary and sufficient:

$$\sigma_i = \sigma_{n-i} \quad \text{for } 0 \le i \le n \tag{5.20}$$

We define a quadratic curve and let $\sigma_0 = \sigma_2$. We define a Bézier-σ curve as

$$\rho(t)^\alpha = (1-t)^2\sigma_0 + 2(1-t)t\sigma_1 + t^2\sigma_2 \tag{5.21}$$

where $t = s/L$. ρ^α is given by a symmetrical quadratic graph of parameter t and the curve itself is symmetric with G^∞. Note that because of symmetry we inevitably lose curvature monotonicity.

Figure 4 shows several Bézier σ-curve with various α values. The start point of each curve is located at the origin and the direction angle there is in the horizontal direction. The curves are defined by $L = 1, \sigma_0 = \sigma_2 = 4$ and $\sigma_1 = 1$. α values are changed from 2 to -1. The curves can be changed drastically due to the change of α.

6. Conclusions

In this research, we have proposed two types of fairness metric functionals for fairing a plane curve defined by similarity curvature and similarity radius of curvature, which are invariant in similarity geometry. We have shown that by minimizing the integral of square of similarity curvature, we obtain LACs whose α equals to 1/2. Similarly for similarity radius of curvature, we obtain LACs for α equals to 3/2. Thus, a clear interpretation of the effect of the slope of the logarithmic curvature graph, especially when α is equal to 1/2 and 3/2 are derived.

We have extended our functionals to handle general LACs by introducing a power function of similarity curvature. The new functionals defined by similarity geometry invariants in Eqn. (4.15) is remarkably better than those previously proposed in [5, 10] because of its scale invariance. Furthermore, we have extended LAC by modifying the integrand of the functional and obtained quasi aesthetic curves and proposed σ curve to introduce symmetry concept for LAC. For future work, we would like to clarify the relationship between the quasi aesthetic and σ-curves and to extend our fairing metrics for free-form surfaces.

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