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Regularized Set Operations in Solid Modeling Revisited

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Abstract. The seminal works of Tilove and Requicha first introduced the use of general topology concepts in solid modeling. However, their works are brute-force approach from mathematicians' perspective and are not easy to be comprehended by engineering trained personnels. This paper reviewed the point set topological approach using operational formulation and provided an alternate method to check the correctness of regularized set operations that was difficult to be formulated or ignored in various research work or literature study of geometric modeling. The results simply provide an alternate approach that easier to check the correctness for future research and development in geometric modeling. Essential topological concepts are described and visualized in easy to understand directed graphs. These facilitate subtle differentiation of key concepts to be recognized. In particular, some results are restated in a more mathematically correct version. Examples to use the prefix unary operators are demonstrated in solid modeling properties derivations and proofs as well as engineering applications.

Keywords: Regularised Set Operations, Constructive Solid Geometry, Solid Modeling, Manufacturing, General Topology.
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1 INTRODUCTION

The seminal works of Tilove and Requicha [9],[14],[15] first introduced the use of general topology concepts in solid modeling. However, their works are brute-force approach from mathematicians' perspective and are not easy to be comprehended by engineering trained personnels. This paper reviewed the point set topological approach using operational formulation. Essential topological concepts are described and visualized in easy to understand directed graphs. These facilitate subtle differentiation of key concepts to be recognized. In particular, some results are restated in a more mathematically correct version. Examples to use the prefix unary operators are demonstrated in solid modeling properties derivations and proofs as well as engineering applications.

This paper is motivated by confusion in the use of mathematical terms like boundedness, boundary, interior, open set, exterior, complement, closed set, closure, regular set, etc. Hasse

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diagrams are drawn by mathematicians to study different topologies. Instead, directed graph using elementary operators of closure and complement are drawn to show their inter-relationship in the Euclidean three-dimensional space ($\mathbb{E}^3$ is a special metric space formed from Cartesian product $\mathbb{R}^3$ with Euclid's postulates [12] and inheriting all general topology properties.) In mathematics, regular, regular open and regular closed are similar but different concepts. The latter is the more correct term for Tilove and Requicha's works. Regular closed set also suffers the limitation that it can be bounded or unbounded. In solid modeling, valid solid is only a subset of compact (closed and bounded) set. However, no mathematical definition for valid solid can be found.

In order to maneuver properties in solid modeling, interior operator is more convenient. The operation approach is useful to derive the inclusion properties of dangling or pendant boundary and "open" boundary situations in traditional set intersection and difference respectively. Mathematically, arbitrary set in Euclidean three-dimensional space can be open, closed, clopen, and neither open nor closed, as well as with or without irregular boundary. Other than "open" and dangling (also called pendant) boundary, detached boundary and isolated points are also possible in general topology. As a result, regular closed set is verified to be necessary in constructive solid geometry representation. Many properties are also found to be more correctly stated as subset inclusion identities rather than equalities.

1.1 Constructive Solid Geometry (CSG)

Nowadays, with the aids of advanced computer-aided design (CAD) modelling tools, various applications such as advanced engineering product design and analysis, virtual reality (VR) and augmented reality (AR) visualizations or 3D printing of complicated physical components in different disciplines are rapidly developed and implemented [5],[13]. In 3D model representation, solid modelling is normally employed and provide methods to overcome the limitations of wireframe and surface modelling [2],[8],[11]. The wire frame and surface modelling approaches have limited engineering applications due to insufficient of topological description and incompleteness in the geometric information. The use of solid modelling method can allow designers to create precise solid models with the aid of Finite Element Analysis (FEA) under a simulated environment.

Accurate 3D complex models or assemblies can be designed and created by solid modelling. Besides, solid modelling can be employed to assess and evaluate the performances such as size, dimension, shape, functionality, or material utilization of complex products or assemblies during preliminary conceptual design stage. To implement the slicing of CAD models, Huang et al. [6] present a robust and efficient approach to directly slicing implicit solids with the proofs for the correctness. The proposed technique allows good distortion error control on the generated contours. Correct objects can be implemented and fabricated by rapid prototyping.

CSG approach [1], [17] is one of the most popular solid representation methods with, user-friendly, accuracy, and validity. Nowadays, various CAD modeling tools uses CSG approach for product design and visualization, e.g. TinkerCAD [16]. A CSG model assumed that physical objected can be represented as a combination of simpler solid primitives. The primitives are cube, cylinder, cone, sphere, torus, etc. Instances of such primitive shapes are created. A complete solid model or assembly is created by combining these objects by Boolean Operations – union, difference, intersection. Boolean operations [10] are better modelling technique in 3D modelling systems that CSG allow users to create and modify models with convenient and easy editing capability among various representation schemes.

2 MATHEMATICAL PRELIMINARIES

This section highlights mathematical concepts essential to solid modeling. In 1922 Kuratowski reported in his breakthrough paper a maximum of fourteen distinct sets by alternate application of complement and closure to any set [3-4],[7].

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2.1 Definitions

The complement of arbitrary set A, denoted cA, is simply points does not belong to A. In other words, if $\mathbb{U}$ is the universal set, then $cA = \mathbb{U}\text{-}A$. Closure can be defined in many ways. Relevant to solid modeling, the closure of A, denoted kA, is the disjoint union of the set of all isolated points of A and the set of all limit points of A. p $\in$ A is an isolated point of A, if and only if there exists an open neighborhood of p which does not contain any other points of A. A point p $\in \mathbb{U}$ is a limit point (also called accumulation point or cluster point) of A if every open neighborhood of p contains another distinct point q $\neq$ p such that q $\in$ A.

A point which has an open ball completely contained in A is an interior point of A. From these come three important definitions. The interior of a set A is the complement of the closure of the complement of A, $iA=ckcA$. The exterior of a set A is the complement of the closure of A, $eA=ckA$. The boundary (also called frontier) of a set A is the intersection of its closure and the closure of its complement, e.g. Equation (2.1):

$$\partial A = kA \cap kcA \tag{2.1}$$

Obviously, $\partial A=\partial cA$ can be expressed as below Eqn. (2.2):

$$\partial cA = kcA \cap kccA \tag{2.2}$$

To the layman, the boundary is considered as the set without its interior, i.e. $A\text{-}iA$. Confusion arises as shown in Equation (2.3) below is not equaled to ∂A but only its subset.

$$A\text{-}iA = A \cap ciA = A \cap kcA \tag{2.3}$$

Figure 1 below shows an arbitrary set which is neither open nor closed, and with isolated points belonging to A and cA. Note that limit points can be interior points or boundary points. Isolated points are boundary points. Also, ∂A or ∂cA includes all boundary limit points and isolated points common to both A and cA.

It should also be noted that kA includes boundary limit points of cA and irregular boundary points of cA are absorbed into limit points of A. Clarification is needed as sets can be open, closed, clopen (a portmanteau of closed-open), and neither open nor closed. A set is closed iff it contains all of its boundary points, i.e. $A=kA$. A set is open iff it contains all of its interior points, i.e. $A=iA$.

In topology, a clopen set in a topological space is a set which is both open and closed. In the Euclidean three-dimensional space of solid modeling, the only two clopen sets are the universal set $\mathbb{U} = \mathbb{E}^3$ and the empty set, e.g. Equation (2.4) and Equation (2.5):

$$k\mathbb{E}^3 = i\mathbb{E}^3 = \mathbb{E}^3 \tag{2.4}$$

$$k\emptyset = i\,\emptyset = \emptyset. \tag{2.5}$$

Instead of Hasse diagram, the inter-relationships of Kuratowski's 14 sets are explicitly depicted as directed graphs. The legends are: prefix unary operators: c (complement), k (closure), i (interior), e (exterior), ∂ (boundary). Sets boxed in solid line are closed, in dashed line are open, and unboxed are undecided. Directed lines refer to mappings of respective operations. Fourteen distinct sets plus boundary created by closure and complement operators for arbitrary set are shown by Figure 2.

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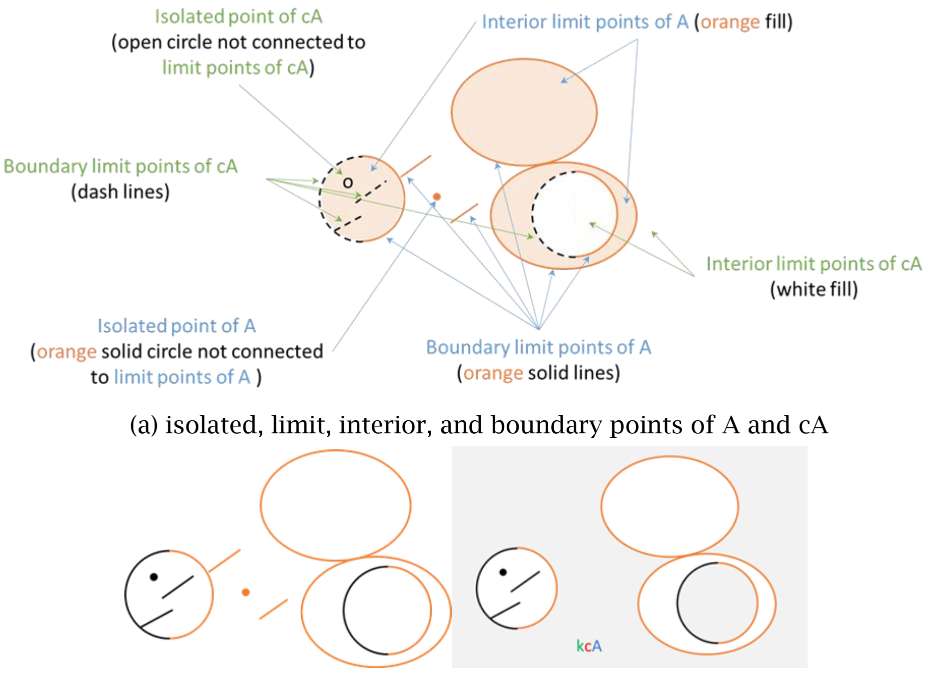

(a) isolated, limit, interior, and boundary points of A and cA

(b) $\partial A = \partial cA$ (c) kcA

Figure 1: Differentiation of points of A and cA.

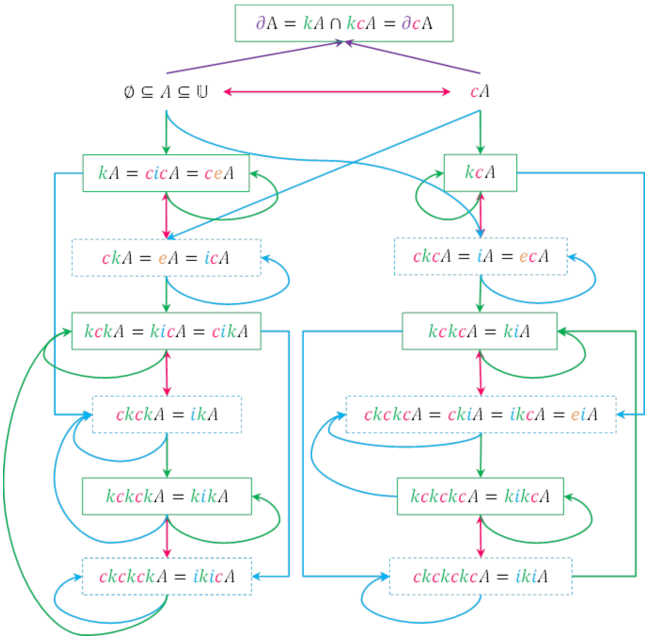

Figure 2: Fourteen distinct sets plus boundary created by closure and complement operators for arbitrary set.

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The subset-superset relationship for arbitrary set is illustrated by Figure 3 below:

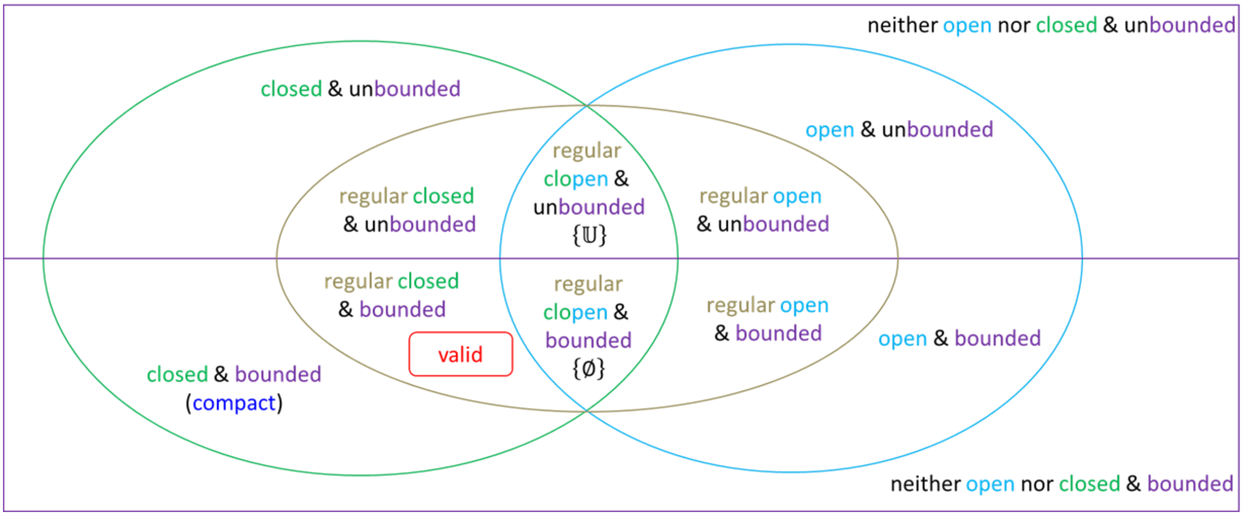

Figure 3: Subset-superset relationship for arbitrary set.

Chapman and Stalnaker proved that Kuratowski's 14 sets can be grouped into 7 pairs (for *A* and *cA*) using closure and interior operations [3],[4] and their relationship is illustrated by Figure 4:

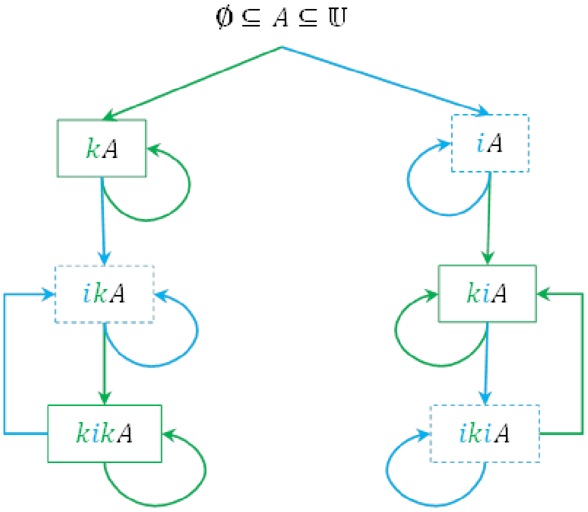

Figure 4: Seven distinct sets created by closure and interior operators for arbitrary set (and its complement).

From Kuratowski's expansive and shrinking relationships [7] of subset containment between an arbitrary set with its closure and interior respectively, e.g. Equation (2.6):

$$iA \subseteq A \subseteq kA \tag{2.6}$$

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Further relations can be found for other sets of Chapman and Stalnaker as Equation (2.7):

$$iA \subseteq ikiA \subseteq \begin{matrix} kiA \\ ikA \end{matrix} \subseteq kikA \subseteq kA \tag{2.7}$$

The above is equivalent to, e.g. Equations (2.8) and (2.9):

$$iA \subseteq ikiA \subseteq ikA \subseteq kikA \subseteq kA \tag{2.8}$$

$$iA \subseteq ikiA \subseteq kiaA \subseteq kikA \subseteq kA \tag{2.9}$$

It should be noted that, for general arbitrary set, there is no explicit relationship between Equations (2.10), (2.11), and (2.12):

$$A \ \& \ kiA \tag{2.10}$$

$$A \ \& \ ikA \tag{2.11}$$

$$ikA \ \& \ kiA \tag{2.12}$$

For special situations,

- a set is called **regular closed** if it is equal to the closure of the interior of itself, i.e. $A = kiA$.
- a set is called **regular open** if it is equal to the interior of the closure of itself, i.e. $A = ikA$.

In addition, boundedness is a different concept. All the fourteen sets can be bounded or unbounded depending on the metrics and the universal set defining the metric space. The boundary is always bounded.

In solid modeling, the topology $\mathbb{U} = \mathbb{E}^3$ being used is connected as according definition, the only clopen set is the universal set and the empty set. Sets obtained by different combinations of closure and interior operations for special sets is shown in Table 1 below:

	$A = iA$	$A = ikiA$	$A = ikA$	$A = kiA$	$A = kikA$	$A = kA$
$iA =$	iA	$ikiA$	ikA	$ikiA$	ikA	ikA
$ikiA =$	$ikiA$	$ikiA$	ikA	$ikiA$	ikA	ikA
$ikA =$	$ikiA$	$ikiA$	ikA	$ikiA$	ikA	ikA
$kiA =$	kiA	kiA	$kikA$	kiA	$kikA$	$kikA$
$kikA =$	kiA	kiA	$kikA$	kiA	$kikA$	$kikA$
$kA =$	kiA	kiA	$kikA$	kiA	$kikA$	kA

Table 1: Sets obtained by different combinations of closure and interior operators for special sets.

Besides, shaded boxes indicate idempotent properties is listed in Table 2.

	c	k	i	∂	e
c	1	$ck = e$	$ci = kc$	$c\partial = e\partial$	$ce = k$

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k	$kc = ci$	$kk = k$	ki	$k\partial = \partial$	$ke = kck = kic$
i	$ic = e$	ik	$ii = i$	$i\partial$	$ie = e$
∂	$\partial c = \partial$	$\partial k = \partial e$	∂i	$\partial\partial \subseteq \partial$	$\partial e = \partial k$
e	$ec = i$	$ek = e$	$ei = cki = ikc$	$e\partial = c\partial$	$ee = ik$

Table 2: Equivalent operators obtained from different combinations of the five main operators.

Double complement follows involution law, e.g. Equations (2.13) and (2.14)

$$\partial\partial\partial = \partial \tag{2.13}$$

$$\partial\partial A = \partial A \Leftrightarrow A = kA \text{ or } A = iA \tag{2.14}$$

2.2 Properties of Regularized Set Operations

In mathematics and computer science, regularization is the term used to describe the technique of modification in order to solve an ill-posed problem. In solid modeling, regularization is used to rectify irregular boundaries resulted after ordinary set operations. To be more precise, the regularized set operations are affected by interior and then closure operations after ordinary set union, intersection, difference, and complement. Rather than illustrating with meagre examples diagrammatically as found in the literature, the following proofs try to verify algebraically the applicability of the regularized set operations in general. To prove, e.g. Equations (2.15), (2.16) and (2.17):

$$ki(kiA_1 \cup kiA_2) = kiA_1 \cup kiA_2 \tag{2.15}$$

$$ki(kiA_1 \cap kiA_2) \subseteq kiA_1 \cap kiA_2 \tag{2.16}$$

$$ki(kiA_1 - kiA_2) \neq kiA_1 - kiA_2 \tag{2.17}$$

Proof of Equation (2.15). Finite union of regular closed sets is regular closed e.g. Equation (2.18),

$$ki(kiA_1 \cup kiA_2) \supseteq k(ikiA_1 \cup ikiA_2) = kikiA_1 \cup kikiA_2 = kiA_1 \cup kiA_2 \tag{2.18}$$

But, e.g. Equation (2.19)

$$ki(kiA_1 \cup kiA_2) \subseteq k(kiA_1 \cup kiA_2) = kkiA_1 \cup kkiA_2 = kiA_1 \cup kiA_2 \tag{2.19}$$

Hence, e.g. Equation (2.20)

$$ki(kiA_1 \cup kiA_2) = kiA_1 \cup kiA_2 \ \blacksquare \tag{2.20}$$

Proof of Equation (2.16): Intersection of regular closed sets is not necessarily regular closed. For arbitrary sets A_1 & A_2 e.g. Equation (2.21),

$$ki(kiA_1 \cap kiA_2) = k(ikiA_1 \cap ikiA_2) \subseteq kikiA_1 \cap kikiA_2 = kiA_1 \cap kiA_2 \ \blacksquare \tag{2.21}$$

Comparing interiors, e.g. Equations (2.22), (2.23), and (2.24):

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$$i(kiA_1 \cap kiA_2) \subseteq iki(kiA_1 \cap kiA_2) \tag{2.22}$$

$$iki(kiA_1 \cap kiA_2) = ik(ikiA_1 \cap ikiA_2) \subseteq i(kikiA_1 \cap kikiA_2) = i(kiA_1 \cap kiA_2) \tag{2.23}$$

$$\therefore iki(kiA_1 \cap kiA_2) = i(kiA_1 \cap kiA_2) \ \blacksquare \tag{2.24}$$

Ordinary and regularized intersection has the same interior.

Comparing boundaries, e.g. Equations (2.25), (2.26), and (2.27):

$$\begin{aligned}(kiA_1 \cap kiA_2) - i(kiA_1 \cap kiA_2) &= kiA_1 \cap kiA_2 \cap ci(kiA_1 \cap kiA_2)\\ &= kiA_1 \cap kiA_2 \cap kc(kiA_1 \cap kiA_2)\end{aligned} \tag{2.25}$$

$$\begin{aligned}\partial ki(kiA_1 \cap kiA_2) &= kki(kiA_1 \cap kiA_2) \cap kcki(kiA_1 \cap kiA_2) = k(ikiA_1 \cap ikiA_2) \cap \\ & kck(ikiA_1 \cap ikiA_2) \subseteq kikiA_1 \cap kikiA_2 \cap kc(kikiA_1 \cap kikiA_2) = kiA_1 \cap kiA_2 \cap \\ & kc(kiA \cap kiA_2)\end{aligned} \tag{2.26}$$

$$\therefore \partial ki(kiA_1 \cap kiA_2) \subseteq (kiA_1 \cap kiA_2) - i(kiA_1 \cap kiA_2) \ \blacksquare \tag{2.27}$$

In other words, extra (isolated, detached, dangling or pendant) boundary exists in general and post intersection regular closed operation is needed.

Proof of Equation (2.17):

(i) From Figure 5 below, $kiA_1 - kiA_2$ and $ki(kiA_1 - kiA_2)$ are different (in $\mathbb{E}^3$) in general.

In addition, Equation (2.28) is neither closed nor open!

$$kiA_1 - kiA_2 = kiA_1 \cap ckiA_2 = kiA_1 \cap ikcA_2 \tag{2.28}$$

As shown in Figure 5, it may have isolated, pendant and detached boundary points, and limit points belonging to its complement. As intersection, irregular boundary may exist and have to be regularized by post difference regular closed operation. Note that $kiA_1 - kiA_2$ and $ki(kiA_1 - kiA_2)$ has the same interior though different boundaries.

(ii) To prove (same interior), e.g. Equation (2.29) below:

$$i(kiA_1 - kiA_2) = iki(kiA_1 - kiA_2) \tag{2.29}$$

Proof: For arbitrary sets e.g. Equation (2.30) below:

$$\begin{aligned}iki(kiA_1 - kiA_2) &= ik(ikiA_1 \cap ckiA_2) \subseteq i(kikiA_1 \cap kckiA_2) = i(kiA_1 \cap kikcA_2) = \\ ikiA_1 \cap ikikcA_2 &= ikiA_1 \cap ikcA_2 = i(kiA_1 \cap ikcA_2) = i(kiA_1 - kiA_2)\end{aligned} \tag{2.30}$$

However, it has Equation (2.31) below:

$$i(kiA_1 - kiA_2) \subseteq iki(kiA_1 - kiA_2) \tag{2.31}$$

Hence, Equation (2.32) is set and listed as below:

$$i(kiA_1 - kiA_2) = iki(kiA_1 - kiA_2) \ \blacksquare \tag{2.32}$$

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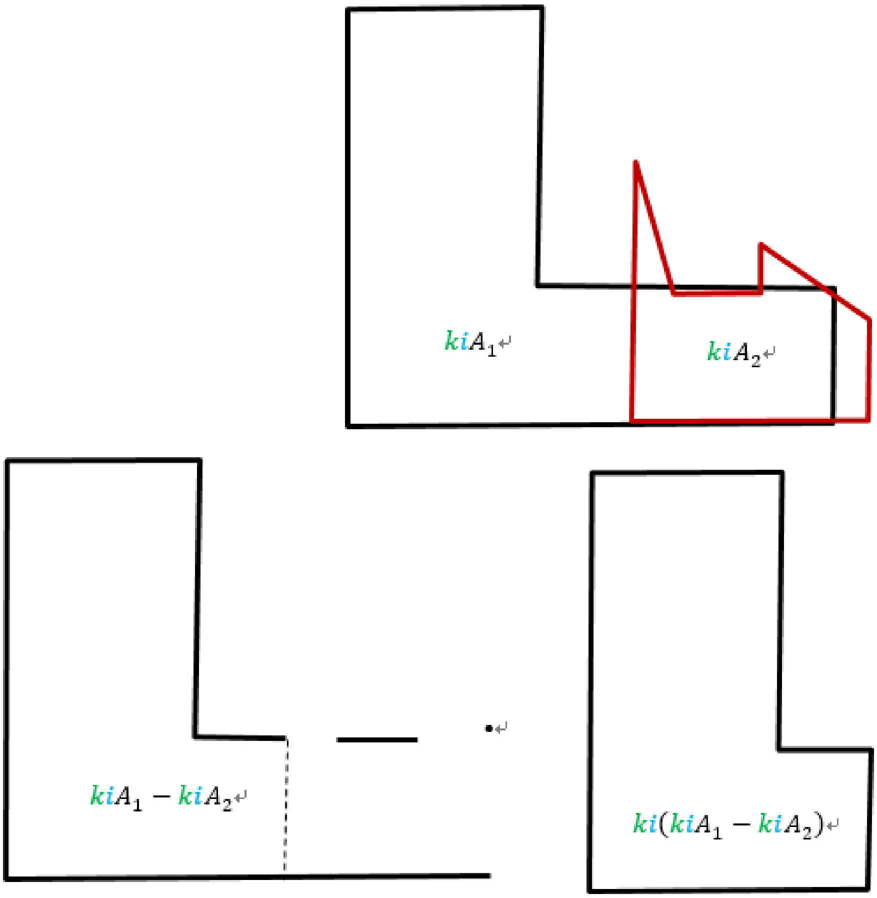

Figure 5: Example to illustrate isolated, detached, dangling & "open" boundaries due to regularized closed difference.

Given the Equation (2.33) below:

$$A_1 \cap A_2 = A_1 - (A_1 - A_2), A_1, A_2 \subseteq \mathbb{U} \tag{2.33}$$

To prove the Equation (2.32), Equation (2.34) is set below:

$$ki(kiA_1 \cap kiA_2) = ki[kiA_1 - ki(kiA_1 - kiA_2)] \tag{2.34}$$

Proof of Equation (2.34), Equations (2.35) and (2.36) is set below:

$$ki[kiA_1 - ki(kiA_1 - kiA_2)] = ki[kiA_1 \cap cki(kiA_1 \cap ckiA_2)] = ki[kiA_1 \cap ikc(kiA_1 \cap ckiA_2)] = ki[kiA_1 \cap ik(ckiA_1 \cup cckiA_2)] = ki[kiA_1 \cap ik(ckiA_1 \cup kiA_2)] = ki[kiA_1 \cap i(kckiA_1 \cup kkiA_2)] = ki[kiA_1 \cap i(kckiA_1 \cup kiA_2)] \supseteq ki[kiA_1 \cap (ikckiA_1 \cup ikiA_2)] = ki[kiA_1 \cap (ckikiA_1 \cup ikiA_2)] = ki[kiA_1 \cap (ckiA_1 \cup ikiA_2)] = ki[(kiA_1 \cap ckiA_1) \cup (kiA_1 \cap ikiA_2)] = ki[\emptyset \cup (kiA_1 \cap ikiA_2)] = k(ikiA_1 \cap iikiA_2) = k(ikiA_1 \cap ikiA_2) = ki(kiA_1 \cap kiA_2) \tag{2.35}$$

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Hence,

$$ki[kiA_1 - ki(kiA_1 - kiA_2)] \supseteq ki(kiA_1 \cap kiA_2) \blacksquare \tag{2.36}$$

As a result, CSG tree cannot be implemented with regularized closed union and difference only. In addition, if the swept solid generated in the process of positive offset [11] of kiA_1 is regular closed kiA_2, then the positive offset result is simply their union. However, to maintain the negative offset result to be regular closed, regularized difference or $ki(kiA_1 - kiA_2)$ should be used.

2.3 Regularized Complement

Ordinary complement has the involution property, e.g. Equation (2.37),

$$ccA = A \tag{2.37}$$

Moreover, complement of closed set is open while complement of open set is closed, e.g. Equations (2.38) and (2.39),

$$c(kA) = ckccA = icA \tag{2.38}$$

and

$$c(iA) = cckcA = kcA \tag{2.39}$$

From the Tilove's work in [14] and [15], regularized complement is defined to be $kic(kiA)$. No explanation nor description on its use is mentioned. The relationship of regularized complement is set by Equation (2.40),

$$kic(kiA) = kckcckckcA = kckckcA = kikcA \tag{2.40}$$

From Table 3, the kic operation satisfies the involution property for regular closed set, e.g. Equation (2.41),

$$(kic)^2(kiA) = kckcckckcckckcA = kckckckcA = kckcA = kiA \tag{2.41}$$

However, the kic operation always gives closed set and will not interchange open set with closed set, e.g. Equations (2.42) and (2.43) below:

$$kic(kA) = kckcckA = kckA = kicA \tag{2.42}$$

and

$$kic(iA) = kckccckcA = kckckcA = kikcA \tag{2.43}$$

Also, it can be proved that the closed complement operations when combined with regularized set operations do not satisfy all de Morgan's law as compared in Table 3.

For regular closed set, there is no irregular boundary. But it may be bounded or unbounded. In Euclidean 3-dimensional space, the universal set $\mathbb{E}^3$ is unbounded ($\mathbb{R} = (-\infty, \infty)$) while the empty set is bounded. In this metric space, the complement of a bounded set will be unbounded and vice versa. As rigid motion in CSG tree (edge) being used to re-position and/or re-orient the leaf and subtree is finite, Table 4 and Figure 6 can then be obtained by assuming orientable boundary. The kic operation will then be useful to ensure all regular closed set to be bounded. Note that the other

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closed complement operators $kikc$ and kc will behave the same as kic. Properties of candidate complements applied on different types of sets is tabulated by Table 5.

de Morgan's law	$c(kiA_1 \cap kiA_2) = ckiA_1 \cup ckiA_2$
	$c(kiA_1 \cup kiA_2) = ckiA_1 \cap ckiA_2$
c	$cki(kiA_1 \cap kiA_2) \neq ki(ckiA_1 \cup ckiA_2)$
	$cki(kiA_1 \cup kiA_2) \neq ki(ckiA_1 \cap ckiA_2)$
kc	$kcki(kiA_1 \cap kiA_2) = ki(kckiA_1 \cup kckiA_2)$
	$kcki(kiA_1 \cup kiA_2) \subseteq ki(kckiA_1 \cap kckiA_2)$
$kikc$	$kikcki(kiA_1 \cap kiA_2) = ki(kikckiA_1 \cup kikckiA_2)$
	$kikcki(kiA_1 \cup kiA_2) \subseteq ki(kikckiA_1 \cap kikckiA_2)$
kic	$kicki(kiA_1 \cap kiA_2) = ki(kickiA_1 \cup kickiA_2)$
	$kicki(kiA_1 \cup kiA_2) \subseteq ki(kickiA_1 \cap kickiA_2)$

Table 3: Different closed complement operations vs. de Morgan's laws.

operand	*operator*	*operand*	*result*
	c	bounded set	unbounded set
	c	unbounded set	bounded set
bounded set	$\cup$	bounded set	bounded set
bounded set	$\cup$	unbounded set	unbounded set
unbounded set	$\cup$	unbounded set	unbounded set
bounded set	$\cap$	bounded set	bounded set
bounded set	$\cap$	unbounded set	bounded set
unbounded set	$\cap$	unbounded set	bounded/unbounded set
bounded set	$-$	bounded set	bounded set
bounded set	$-$	unbounded set	bounded set
unbounded set	$-$	bounded set	bounded/unbounded set
unbounded set	$-$	unbounded set	bounded set

Table 4: Results on operations of bounded and unbounded set.

Figure 6: Intersection of 2 unbounded sets can be bounded or unbounded.

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	A	iA	$ikiA$	ikA	kiA	$kikA$	kA
c	cA	kcA	$kikcA$	$kicA$	$ikcA$	$ikicA$	icA
ic	icA	$ikcA$	$ikcA$	$ikicA$	$ikcA$	$ikicA$	icA
$ikic$	$ikicA$	$ikcA$	$ikcA$	$ikicA$	$ikcA$	$ikicA$	$ikicA$
ikc	$ikcA$	$ikcA$	$ikcA$	$ikicA$	$ikcA$	$ikicA$	$ikicA$
kic	$kicA$	$kikcA$	$kikcA$	$kicA$	$kikcA$	$kicA$	$kicA$
$kikc$	$kikcA$	$kikcA$	$kikcA$	$kicA$	$kikcA$	$kicA$	$kicA$
kc	kcA	kcA	$kikcA$	$kicA$	$kikcA$	$kicA$	$kicA$

(a) candidate complements

	A	iA	$ikiA$	ikA	kiA	$kikA$	kA
$(c)^2 = 1$	A	iA	$ikiA$	ikA	kiA	$kikA$	kA
$(ic)^2 = ik$	ikA	$ikiA$	$ikiA$	ikA	$ikiA$	ikA	ikA
$(ikic)^2 = ik$	ikA	$ikiA$	$ikiA$	ikA	$ikiA$	ikA	ikA
$(ikc)^2 = iki$	$ikiA$	$ikiA$	$ikiA$	ikA	$ikiA$	ikA	ikA
$(kic)^2 = kik$	$kikA$	kiA	kiA	$kikA$	kiA	$kikA$	$kikA$
$(kikc)^2 = ki$	kiA	kiA	kiA	$kikA$	kiA	$kikA$	$kikA$
$(kc)^2 = ki$	kiA	kiA	kiA	$kikA$	kiA	$kikA$	$kikA$

(b) double complements

Table 5: Properties of candidate complements applied on different types of sets.

2.4 Regularized Complement and Negative Offset

According to Rossignac's work [11] on offset operation, denote the (regularized) positive offset, then the (regularized) negative solid offset of a non-empty S is defined as the complement of the expanded complement, e.g. Equation (2.44),

$$S \downarrow^* r = c^*\big((c^*S) \uparrow^* r\big) \tag{2.44}$$

Let kiA be the regular closed set to be offset. kiA_s be the regular closed set obtained by sweeping a closed ball of radius r centered at ∂kiA.

One can formulate positive offset as Equation (2.45):

$$(kiA) \uparrow^* r = kiA \cup kiA_s \tag{2.45}$$

and negative offset as Equation (2.46):

$$(kiA) \downarrow^* r = ki(kiA - kiA_s) \tag{2.46}$$

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The following is to check all negative offset definitions (using regularized difference and different closed complements: kc, $kikc$, kic) are equivalent as Equation (2.47):

$$kic(kickiA \cup kiA_s) = kikc(kikckiA \cup kiA_s) = kc(kckiA \cup kiA_s) = ki(kiA - kiA_s) \tag{2.47}$$

Proof by Equations (2.48) to (2.50):

$$kic(kickiA \cup kiA_s) = kic(kikcA \cup kiA_s) = ki(ckikcA \cap ckiA_s) = ki(ikiA \cap iikcA_s) = ki(kiA \cap ikcA_s) = ki(kiA \cap ckiA_s) = ki(kiA - kiA_s) \blacksquare \tag{2.48}$$

$$kikc(kikckiA \cup kiA_s) = kikc(kikcA \cup kiA_s) = kik(ckikcA \cap ckiA_s) = kik(ikiA \cap ckiA_s) = kik(ikiA \cap iikcA_s) = kiki(kiA \cap ikcA_s) = ki(kiA - kiA_s) \blacksquare \tag{2.49}$$

$$kc(kckiA \cup kiA_s) = k(ckckiA \cap ckiA_s) = k(ikiA \cap iikcA_s) = ki(kiA - kiA_s) \blacksquare \tag{2.50}$$

3 MANUFACTURING PROCESS & REGULAR CLOSED OPERATORS

Manufacturing or fabrication processes can be classified into one of the three types: additive, subtractive, and formative [5]. Whether or not parts are moldable or castable have been studied using visibility. This section discusses how to model parts manufactured in different categories with regular closed operators.

3.1 Additive

Additive includes processes like joining, bonding, welding and glueing as well as interlocking in single laser beam stereolithography and sintering scanning and dual beam curing of liquid photocurable resin.

Input (Parts to be unioned together) is described by Equation (3.1):

$$kiA_1 \& kiA_2 \tag{3.1}$$

Output is described by Equation (3.2):

$$kiA_1 \cup kiA_2 \tag{3.2}$$

Regularized closed operation after union is redundant as Equation (3.3):

$$ki(kiA_1 \cup kiA_2) = kiA_1 \cup kiA_2 \tag{3.3}$$

In addition, result of ACSG (additive CSG) tree [2] after finite number of addition is shown by Equation (3.4):

$$\bigcup_{j=1}^{n} kiA_j \tag{3.4}$$

as regularized union is associative.

Property (regularized union is associative): Given the following Equation (3.5):

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$$kiA_1 \cup (kiA_2 \cup kiA_3) = (kiA_1 \cup kiA_2) \cup kiA_3 \tag{3.5}$$

To prove with an Equation (3.6):

$$ki\big(kiA_1 \cup ki(kiA_2 \cup kiA_3)\big) = ki(ki(kiA_1 \cup kiA_2) \cup kiA_3)$$
(3.6)

Proof (e.g. equation 3.7):

$$LHS = ki\big(kiA_1 \cup ki(kiA_2 \cup kiA_3)\big) = ki\big(kiA_1 \cup (kiA_2 \cup kiA_3)\big) = ki\big((kiA_1 \cup kiA_2) \cup kiA_3\big) = ki(ki(kiA_1 \cup kiA_2) \cup kiA_3) = RHS \quad \blacksquare \tag{3.7}$$

Fastening, riveting and assembly operations with or without relative motion will not merge the mating parts. In such cases, union is not involved, but appropriate rigid motion transforms of the inputs with respect to the result.

3.2 Subtractive:

Input:

Stock or blank kiA_1 and subtractive feature kiA_2

Output:
The following Equation (3.8) is obtained.

$$ki(kiA_1 - kiA_2) \tag{3.8}$$

Regular closed operation after difference is compulsory as (Equation 3.9) and it may not be regular closed in general.

$$kiA_1 - kiA_2 \tag{3.9}$$

In subtractive machine tool classification, the cutting profile is defined in terms of generatrix and directrix sweeping. The subtractive feature modeled as regular closed set can either be (bounded) solid cutter or (bounded or unbounded) half space, generated by orientable swept surface (e.g. in wire cutting).

Property: regularized union and is given by Equations (3.10) and (3.11).

$$kiA_1 - (kiA_2 \cup kiA_3) = (kiA_1 - kiA_2) - kiA_3 \tag{3.10}$$

$$(kiA_1 - kiA_2) \cup kiA_3 = (kiA_1 \cup kiA_3) - (kiA_2 - kiA_3) \tag{3.11}$$

In general, the two set identities in terms of union and difference do not hold for their regularized closed version, e.g. Equations (3.12) and (3.13),

$$ki\big(kiA_1 - ki(kiA_2 \cup kiA_3)\big) \neq ki(ki(kiA_1 - kiA_2) - kiA_3) \tag{3.12}$$

$$ki(ki(kiA_1 - kiA_2) \cup kiA_3) \neq ki\big(ki(kiA_1 \cup kiA_3) - ki(kiA_2 - kiA_3)\big) \tag{3.13}$$

Proof of Equations (3.10) and (3.12) into Equations (3.14) and (3.15):

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$$LHS = ki\big(kiA_1 - ki(kiA_2 \cup kiA_3)\big) = ki\big(kiA_1 - (kiA_2 \cup kiA_3)\big) = ki\big(kiA_1 \cap c(kiA_2 \cup kiA_3)\big) = ki(kiA_1 \cap ckiA_2 \cap ckiA_3) \tag{3.14}$$

$$RHS = ki(ki(kiA_1 - kiA_2) - kiA_3) = ki(ki(kiA_1 \cap ckiA_2) \cap ckiA_3) \neq LHS \quad \blacksquare \tag{3.15}$$

Proof of Equations (3.11) and (3.13) into Equations (3.16) and (3.17):

$$LHS = ki(ki(kiA_1 - kiA_2) \cup kiA_3) = ki(kiA_1 - kiA_2) \cup kiA_3 = ki(kiA_1 \cap ckiA_2) \cup kiA_3 \tag{3.16}$$

$$RHS = ki\big(ki(kiA_1 \cup kiA_3) - ki(kiA_2 - kiA_3)\big) = ki\big((kiA_1 \cup kiA_3) \cap cki(kiA_2 \cap ckiA_3)\big) = ki\big((kiA_1 \cup kiA_3) \cap ikc(kiA_2 \cap ckiA_3)\big) = ki\big((kiA_1 \cup kiA_3) \cap ik(ckiA_2 \cup kiA_3)\big) \neq LHS \tag{3.17}$$

Corollary:
Therefore, CSG tree with only regularized union and difference cannot be reconfigured as ordinary union and difference. The correct formulation of DSG (destructive solid geometry) tree in should be defined as the Equations (3.18).

$$ki(ki(\cdots ki(ki(kiA_1 - kiA_2) - kiA_3) - \cdots) - kiA_n) \tag{3.18}$$

An alternative definition below is simpler but not identical,

$$ki\big(kiA_1 - \cup_{j=2}^{n} kiA_j\big) \tag{3.19}$$

3.3 Formative:

Casting or injection molding and forming:
Input:

Mold halves kiA_1 & kiA_2

Output:

The part is the closure of the complement of the cavity formed by closing the mould halves which is regular closed and is shown by Equation (3.20),

$$kc(kiA_1 \cup kiA_2) \tag{3.20}$$

To prove with Equation (3.21):

$$ki\big(kc(kiA_1 \cup kiA_2)\big) = kc(kiA_1 \cup kiA_2) \tag{3.21}$$

Proof of Equation (3.22):

$$kikc(kiA_1 \cup kiA_2) = kcki(kiA_1 \cup kiA_2) = kc(kiA_1 \cup kiA_2) \tag{3.22}$$

In reverse engineering of orthographic views to obtain the solid model, swept solids are intersected [13]. Since associative property does not hold for regularized intersection, reconfiguration of

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regularized intersection based CSG tree is not possible in general. Assumption had to be made for no irregular boundary being generated in regularized intersection.

Property (regularized intersection is not associative):

Given an Equation (3.23),

$$kiA_1 \cap (kiA_2 \cap kiA_3) = (kiA_1 \cap kiA_2) \cap kiA_3 \tag{3.23}$$

To prove with Equation (3.24):

$$ki\big(kiA_1 \cap ki(kiA_2 \cap kiA_3)\big) \neq ki(ki(kiA_1 \cap kiA_2) \cap kiA_3) \tag{3.24}$$

Proof of Equations (3.25) and (3.26):

$$LHS = ki\big(kiA_1 \cap ki(kiA_2 \cap kiA_3)\big) \subseteq ki\big(kiA_1 \cap (kikiA_2 \cap kikiA_3)\big) = ki\big(kiA_1 \cap (kiA_2 \cap kiA_3)\big) = ki\big((kiA_1 \cap kiA_2) \cap kiA_3\big) \tag{3.25}$$

$$RHS = ki(ki(kiA_1 \cap kiA_2) \cap kiA_3) \subseteq ki\big((kiA_1 \cap kiA_2) \cap kiA_3\big) \tag{3.26}$$

Consequently, it can be concluded that Equations (3.25) and (3.26) are not equal and is shown by Equation (3.27).

$$\therefore LHS \neq RHS \quad \blacksquare \tag{3.27}$$

4 CONCLUSION

This paper reviewed the point set topological approach using operational formulation and provided an alternate method to check the correctness of regularized set operations that was difficult to be previously formulated. An operational approach of general topology that may be more legible to CAD/CAM practitioners has been explained in detail. Some previous ad-hoc results are refined, and some new properties are derived. However, the above-mentioned operators are unable to detect non-manifold interim results in applying regularized set operations. Nevertheless, it is envisaged that the work presented will provide a solid foundation for future development. For instance, geometric modeling may be extended to using spatial reasoning via parallel operators to spatial modal logic. Essential topological concepts are described and visualized in easy to understand directed graphs with restated results in a more mathematically correct version. Examples to use the prefix unary operators are demonstrated in solid modeling properties derivations and proofs as well as engineering applications. The results in this paper proved that regularized intersection cannot be formulated in terms of regularized union and difference as the non-regularized version. Besides, the results simply provided the alternate approach that is easier to check the correctness for future research and development in geometric modeling. Besides, the implementation of directly slicing CSG models for additive manufacturing with the proofs for the correctness in topology after regularization will be explored for the future work.

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