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Interpolation of Point Sequences with Extremum of Curvature by Log-aesthetic Curves with G^2 continuity

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Abstract. Yan et al. proposed a new family of curves called κ-curves which is composed of quadratic Bézier splines that interpolates a sequence of control points by letting curvatures extrema to occur at control points. Their interpolation method guarantees G^2 continuity at control points and G^1 continuity at inflection points. Recent progress in aesthetic design include the development of Log-Aesthetic (LA) curves, which has been highlighted as the most promising curve due to its special feature of self-affinity and preserving monotonic curvature profiles. In this paper, we propose a novel method to generate LA curves interpolating control points and enforcing local curvatures extrema to occur only at control points while preserving G^2 continuity everywhere, including at inflection points. In addtion, LA curve offers extra degree of freedom; its shape parameter α which can deform the curve according to designer's preference.

Keywords: log-aesthetic Curve, κ-curve, extremum of curvature, G^2 conituity, point sequences
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1 INTRODUCTION

Yan et al. [10] proposed a new family of curves called κ-curves which is composed of quadratic Bézier splines that interpolates a sequence of control points by letting local maxima and minima of curvatures occur at control points. Figure 1 shows an example of κ-curves used to draw the outline for a bear. Their interpolation method guarantees G^2 continuity at control points and G^1 continuity at inflection points. This interpolating curve has caught much attention among researchers as well as designers, and currently available in Adobe Illustrator for practical design. Chen et al. [1] follow suit κ-curve and proposed feature points controlled interpolatory curve (FPC-curve) using piecewise cubic curves to control over geometric features namely cusps, loops and inflection points, which are considered intrinsic features of curves.

One of the disadvantage of quadratic and cubic curves are its natural characteristics of having complex curvature function, hence hard to obtain a monotonic curvature profile segment which is essential for aesthetic design. Recently, Log-aesthetic (LA) curves are claimed to be a promising curve for aesthetic design due to its special feature of preserving monotonic curvature profiles [4] . Yet another essential feature LA curves possess is self-affinity which tend to be useful for satisfying geometry continuity using affine transformation procedures. In this paper, we substitute LA curves for Bézier curves to produce interpolating LA curves, mimicking k-curves. Thus, we would be able to control extrama points on the curve which are now the control points itself.

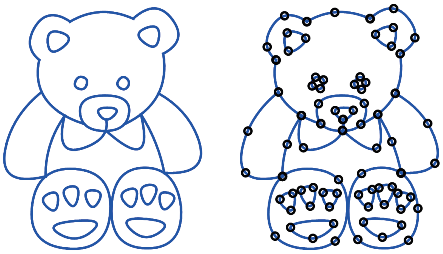

Figure 1: An example of a κ-curve[10].

2 GENERAL EQUATIONS OF AESTHETIC CURVES

LA curves are derived by representing Logarithmic Curvature Graph (LCG) as a straight line (linear LCG). The equation for LCG which has a slope, α, is the fundamental equation of LA curves [6]:

$$\log(\rho\frac{ds}{d\rho}) = \alpha \log \rho + C \tag{1}$$

where s is the arc length of the curve, ρ is the radius of curvature and C is the constant. The shape parameter of the LA curve is α. In this paper we let $\alpha < 0$ in order to generate C-shaped and S-shaped (with inflection point) LA curves interpolating given points.

2.1 Formulas in Standard Form II

The following equation is obtained by substituting $\Lambda = e^{-C}$ into Eq.(1) where $\Lambda \in [0, \infty]$,

$$\frac{ds}{d\rho} = \frac{\rho^{\alpha-1}}{\Lambda} \tag{2}$$

By assuming $\rho(0) = 1$, integrating Eq.(2) and rewriting ρ in terms of arc length s:

$$\rho = (\Lambda\alpha s + 1)^{\frac{1}{\alpha}} \tag{3}$$

which is called standard form II [11]. In this form, ρ is assumed to vary from 0 to ∞, hence the curvature $\kappa = 1/\rho$ also varies from 0 to ∞. ρ can also be expressed in terms of direction angle $\theta(s)$ by using Eq.(2) and $d\theta(s)/ds = 1/\rho$, we obtain:

$$\rho = ((\alpha - 1)\Lambda\theta(s) + 1)^{\frac{1}{\alpha-1}} \tag{4}$$

The arc length s and $\theta(s)$ have various upper and lower bounds, depending on α value. The lower bounds of s and $\theta(s)$ are $-\infty$ and the upper bound of s and $\theta(s)$ are $-1/(\Lambda\alpha)$ and $1/((1-\alpha)\Lambda)$ respectively. Readers are referred to [11] for a detailed study LA inflection points.

Two versions of LA curves with respect to parameter s and θ is given as $\boldsymbol{C}(s)$ and $\boldsymbol{P}(\theta)$:

$$\boldsymbol{C}(s) = \int_0^s e^{i(\frac{(\Lambda\alpha u+1)^{\frac{\alpha-1}{\alpha}}}{\Lambda(\alpha-1)})} du \tag{5}$$

$$\boldsymbol{P}(\theta) = \int_0^\theta ((\alpha-1)\lambda\psi+1)^{\frac{1}{\alpha-1}} e^{i\psi} d\psi \tag{6}$$

2.2 General Case

In this section, we discuss about the LA curves whose $\rho \neq 1$ at $s=0$. The standard form II is used to generate an LA segment and the input control points should be reflected along a line connecting two control points and/or the direction of the curve should be reversed if necessary. For a general case, we let $\alpha < 0$ and the signed curvature [2] of the LA curve, $\kappa(s)$ is defined as follows [5]:

$$\kappa(s) = \begin{cases} (cs+d)^{-\frac{1}{\alpha}} & \text{if } cs+d \geq 0 \\ -(-cs-d)^{-\frac{1}{\alpha}} & \text{otherwise} \end{cases} \tag{7}$$

When $\kappa(s)=0$, we may solve $cs+d=0$, to get $s=-d/c$, hence the point at $s=-d/c$ is an inflection point. Note that at an inflection point, the radius of curvature $\rho \to \infty$, so numerical calculation using ρ with arc length s as shown in Eq.(5) is unstable. Furthermore it is inadequate to use Eq.(5) even though the interval of integration can be separated from 0 to ϕ_{max} and ϕ_{max} to ϕ_D to get the end point of the curve.

The derivative of κ with respect to s is given by

$$\frac{d\kappa}{ds} = \begin{cases} -\frac{c}{\alpha}(cs+d)^{-\frac{1}{\alpha}-1}, & \text{if } cs+d \geq 0 \\ \frac{c}{\alpha}(-cs-d)^{-\frac{1}{\alpha}-1}, & \text{otherwise} \end{cases} \tag{8}$$

If $(-1/\alpha-1) \geq 0$, $d\kappa/ds$ is continuous at the inflection point. However $(-1/\alpha-1) < 0$, i.e. $\alpha < -1$, $d\kappa/ds$ is discontinuous there. Similarly $(-1/\alpha-n) < 0$, i.e. when $\alpha < -1/n$, the n^{th} derivative is discontinuous at the inflection point. The curve $\boldsymbol{C}(s)$ is given by

$$\boldsymbol{C}(s) = \int_0^s e^{i(\theta_s+\frac{\alpha}{c(\alpha-1)}(|cs+d|^{\frac{\alpha-1}{\alpha}}-|d|^{\frac{\alpha-1}{\alpha}}))} du \tag{9}$$

where $\theta(0)=\theta_s$. To generate a segment of LA curve using standard form II, we set the start point $\boldsymbol{P}_s$, the start direction angle θ_s, α, Λ, the total length s_0, scaling factor s_f, reflection flag and reverse flag. Similarly, in general case, we need $\boldsymbol{P}_s$, the start direction angle θ_s and α as in standard form II. In addition, we need c, d, the total length h_0. If we adopt the general case, neither the reflection nor reverse flags are necessary. For implementation of the curve generation, it is sometimes necessary to convert from standard form II to general case in accord to data structure. The parameters of general case are given by

$$c = \lambda\alpha s_f^{\alpha-1} \tag{10}$$

$$d = s_f^{\alpha} \tag{11}$$

$$h_0 = s_f s_0 \tag{12}$$

If the reflection flag is on, then both of the signs of c and d are changed as follows:

$$\hat{c} = -c \tag{13}$$

$$\hat{d} = -d \tag{14}$$

Furthermore if the reverse flag is on, then c remains the same, but

$$\hat{d} = -(d + c\,h_0) \tag{15}$$

3 S-SHAPED CURVE GENERATION

At first we review the method proposed in [5] to generate an S-shaped LA curve as shown in Fig.2. Here we used standard form II as the representaion of the LA curve. The arc length s of a curve with $\alpha < 0$ is given as a function of the directional angle ϕ at the end point as follows:

$$s(\phi) = \frac{\{1 + (\alpha - 1)\Lambda\phi\}^{\frac{\alpha}{\alpha-1}} - 1}{\Lambda\alpha} \tag{16}$$

It is necessary for the total arc length s to become larger than the arc length to the inflection point s_0 to form an S-shaped curve where

$$s_0 = -\frac{1}{\Lambda\alpha} \tag{17}$$

Notice that $s_0 > 0$ due to $\alpha < 0$. If the directional angle is defined to be negative when it decreases over $0°$, the directional angle becomes maximum at the arc length s_0 and is given by

$$\phi_{max} = \frac{1}{(1-\alpha)\Lambda} \tag{18}$$

If the directional angle is specified to be less than the value stated above, it is not possible to generate a curve without a loop which indicates that the directional angle changes by more than 2π. s is expressed by means of s_0 from Eq.(16) as follows:

$$s(\phi) = s_0 + \frac{\{1 + (\alpha - 1)\Lambda\phi\}^{\frac{\alpha}{\alpha-1}}}{\Lambda\alpha} \tag{19}$$

As discussed in the previous section, at the inflection point the continuity of $d^n\kappa/ds^n$ depends on shape parameter α. If we assume that the curve is S-shaped, the second term of Eq.(19) increases according to the increase of $\phi \to \phi_{max}$. Beyond ϕ_{max} the directional angle (ϕ_D in Fig. 2.) decreases. ϕ_D indicates the directional angle of P_e calculated anticlockwise from x axis and, ϕ_D is the angle between the line through the origin and P_e and the x axis. Hence we change the sign of the second term and define s_1 by

$$s_1 = -\frac{\{1 + (\alpha - 1)\Lambda\phi\}^{\frac{\alpha}{\alpha-1}}}{\Lambda\alpha} \tag{20}$$

Note that since $\alpha < 0$, $s_0 > 0$ hence, $s_1 > 0$. The whole total arc length s is given by $s_0 + s_1$.

3.1 C-shaped or S-Shaped?

In this section, we construct a simple theorem to clearly identify whether a generated LA curve from given boundary condition has inflection points which generates S-shaped curves. The following theorem addresses for a given boundary condition , the generated LA curve consists of inflection points, i.e., if $\exists b$ such that $0 < b < h_0$, where $\kappa(b) = d\theta(b)/ds = 0$.

Definition 1. *A boundary condition (as shown in Fig.3) consists of a start point $\boldsymbol{P}_s$ and the end point $\boldsymbol{P}_e$ are located on the x axis and the direction angles at $\boldsymbol{P}_s$ is $\theta_s \in [-\frac{\pi}{2}, 0]$, the direction angle at $\boldsymbol{P}_e$ is $\theta_e \in [-\frac{\pi}{2}, 0]$.*

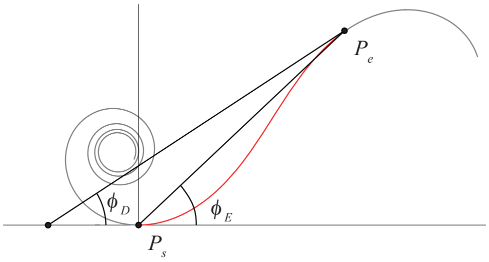

Figure 2: Input of an S-shaped LA curve

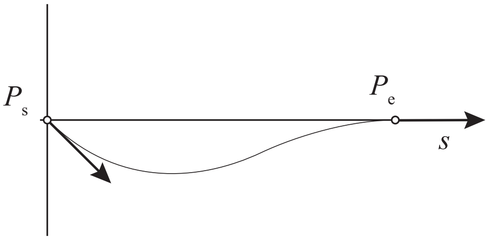

Figure 3: Boundary condition: the start and end points are transformed to the x axis and the end angles are restricted to $\theta_s \in [-\frac{\pi}{2}, 0]$ and $\theta_e[-\frac{\pi}{2}, 0]$. The figure shows for the case with $\theta_e = 0$.

Theorem 1. *Let a LA curve is given as a function of arc length s, $\boldsymbol{C}(s)$, satisfies boundary condition and at least C^1-continuous and it is loop-free. The direction angle function of the curve is defined as $\phi(s) \in [-\pi/2, \pi/2]$.The LA curve which interpolates $\boldsymbol{P}_s = \boldsymbol{C}(0)$, $\boldsymbol{P}_e = \boldsymbol{C}(h_0)$ has an inflection point in $s \in [0, h_0]$.*

Proof. Since the y coordinates of the start and end points are equal to 0, the following equation is satisfied:

$$f(h_0) = \int_0^{h_0} \sin\phi(s)ds = 0 \tag{21}$$

Since $f(0) = f(h_0) = 0$ and $\sin\phi(s)$ is a continuous function, by the mean-value theorem there is a such that $0 < a < h_0$ and $\sin\theta(a) = 0$. Since we assume that $\phi(s) \in [-\pi/2, \pi/2]$, we obtain $\phi(a) = 0$.

When $\theta_e = 0$, by applying the mean value theorem to $\phi(a) = \phi(h_0) = 0$, there exists b such that

$$\frac{d\theta(b)}{ds} = \kappa(b) = 0 \tag{22}$$

where $a < b < h_0$.

When $\theta_e < 0$, as $\sin\phi$ is continuous and $\sin\theta_s < 0$, there exists $\epsilon_s > 0$ and $f(\epsilon_s) < 0$, and as $\sin\theta_e < 0$, there exists $\epsilon_e > 0$ and $f(h - \epsilon_e) > 0$. Hence by the intermediate value theorem, there is c such that $\epsilon_s < c < h_0 - \epsilon_e$ and $f(c) = 0$. By applying the mean value theorem for intervals $[0, c]$ and $[c, h_0]$ of $f(s)$, there is b such that $0 < a < c$, $c < b < h_0$ and $\phi(a) = \phi(b) = 0$. Therefore there exists d such that

$$\frac{d\theta(d)}{ds} = \kappa(d) = 0 \tag{23}$$

where $a < d < b$. Hence the LA curve $\boldsymbol{C}(s)$ has an inflection point at $s^* = b$. □

By using the above theorem we know the generated curve must be a S-shaped curve. However even if $\theta_e > 0$, the curve might be a S-shaped curve as well. Since Miura et al. [5] did not specify the exact condition for θ_e, we describe a numerical method to determine whether the LA curve is C-shaped or S-shaped in this section. Based on the generation method proposed by Yoshida and Saito [11] as shown in Fig.2, if $\theta_D > 2\theta_E$, the start and end points are flipped to obtain $\theta_D \leq 2\theta_E$ and the signs of their direction angles are changed. If $\theta_D = 2\theta_E$, the generated curve becomes a circular arc due to the symmetry of the boundary condition, producing a C-shaped curve. By Theorem 1, if $\theta_D \leq \theta_E$, a S-shaped curve is generated.

In $\theta_D \in [\theta_E/2, \theta_E]$, there is $\exists\theta'_D$ where if $\theta_D < \theta'_D$, the curve is S-shaped and C-shaped otherwise. When the arc length $s < s_0$ in Eq.(17), the angle between the line connecting the point $\boldsymbol{C}(s)$, the origin and the x axis increases as s increases. According to Eq.(17), the larger Λ, the smaller s_0. Also from Eq.(18), for a given θ_D, the maximum value of Λ is

$$\Lambda_{max} = \frac{1}{(1-\alpha)\theta_D} \tag{24}$$

Hence when $s = -1/(\Lambda_{max}\alpha)$, if the angle between the line connecting the point $\boldsymbol{C}(s)$, the origin and the x axis is less than θ_E, then the curve is S-shaped and C-shaped otherewise.

Fig. 4 shows several LA curves generated for three types boundary conditions and their corresponding Λ versus ϕ plotted below them. The shape parameter α is fixed as $\alpha = -0.5$.The left and middle curves are C-shaped LA curves, hence inflection free. In their Λ versus ϕ graphs, the red line indicates $\phi_E = \pi/6$ and the blue curve indicates the direction angle ϕ between the line connecting the point $\boldsymbol{C}(s)$, the origin and the x axis for a given Λ. Λ can change from 0 to $\Lambda_{max} = 1/((1-\alpha)\theta_D)$. In the left and middle graphs, the red line and blue curve are intersects, i.e. when $\Lambda = \Lambda_{max}$, angle ϕ is larger than θ_E. According to our numerical identification above, it means the generated curve is C-shaped, which is consistent in this example. On the other hand, the right curve has an inflection point indicated by $*$ and is S-shaped. When $\Lambda = \Lambda_{max}$, angle ϕ is smaller than θ_E, then we get is S-shaped LA curve.

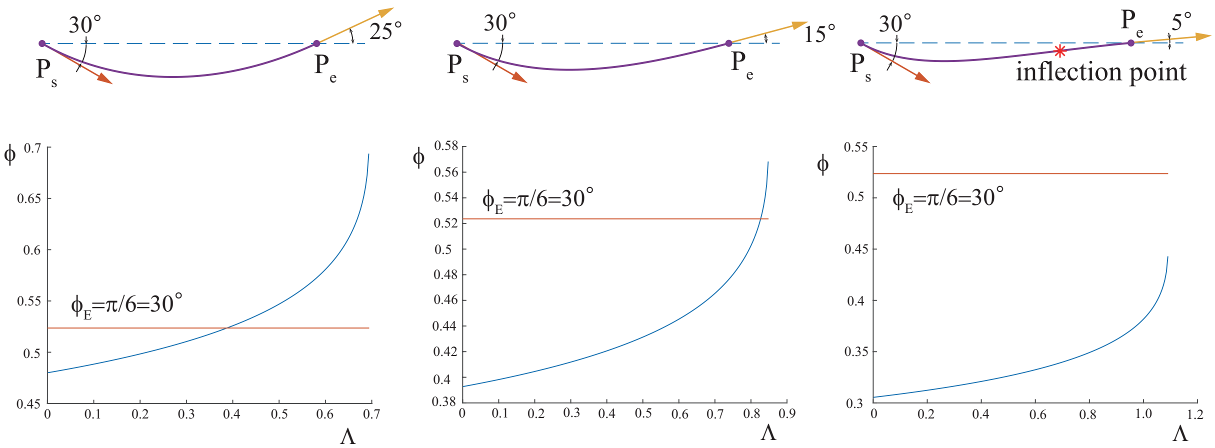

Figure 4: Three boundary conditions: the direction angle at the end point $\boldsymbol{P}_e$ is changed at $25°$, $15°$ and $5°$ and their respective Λ versus ϕ graphs. $(*)$ indicates an inflection point.

Figure 5 shows three types of LA curves generated with $\alpha \in \{-0.5, 2.5, 5\}$. The curve on the left is a C-shaped curve, whereas the other two are S-shaped curves. When $\Lambda = \Lambda_{max}$, the angle $\phi > \theta_E$, the curve is C-shaped. Whereas if the angle $\phi < \theta_E$, the curve is S-shaped, which is also consistent to our C- or S-shaped numerical identification.

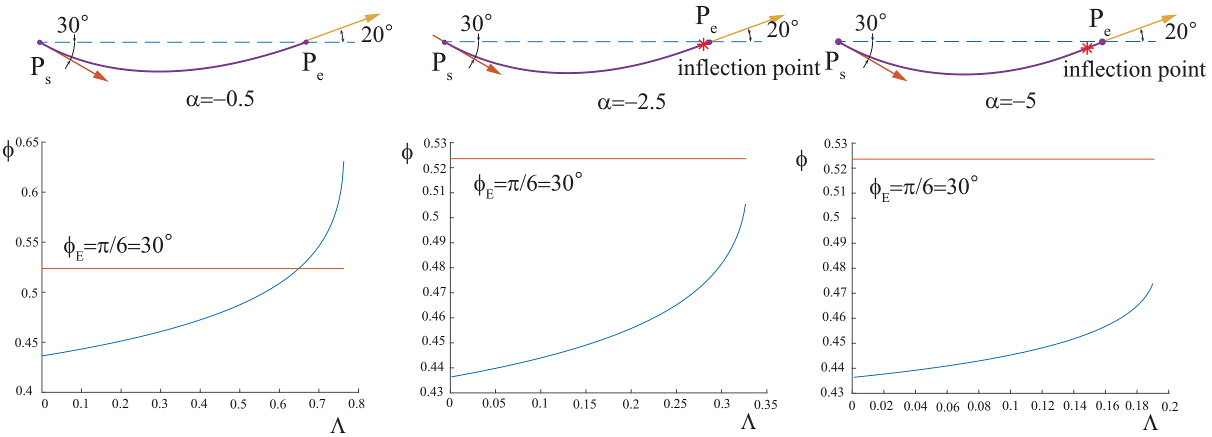

Figure 5: Similar boundary conditions but with three types of shape parameter $\alpha \in \{-0.5, 2.5, 5\}$ and their respective Λ versus ϕ graphs. $(*)$ indicates an inflection point.

4 LOG-AESTHETIC CURVE GENERATION ALGORITHM

As shown in Fig.6, we generate a sequence of LA curves with G^2 continuity by the following algorithm:

1. Input a sequence of control points P_i.
2. Generate κ-curves interpolating the control points $\boldsymbol{P}_i$.
3. Calculate the tangential angle of κ-curves at $\boldsymbol{P}_i$ as initial value of tangent angle θ_{ei} of LA curves.
4. Calculate the line length l_i between control points P_i, the inner angle of the control polygon θ_i, so another angle between the tangential line and the connection line is $\theta_{fi} = \pi - \theta_i - \theta_{ei}$.
5. Generate LA curves with a specified α and a triangle located at the origin and congruent to the triangle defined by the control points.
6. Find Λ from Eq.(3) that satisfies the angles of the triangle using bisection method.
7. Translate, rotate and scale the LA curve to the triangle defined by the control points.
8. Calculate the curvature $\kappa_{i,1}, \kappa_{i+1,0}$ of the LA curves at the control points $\boldsymbol{P}_i$.
9. Compare the curvature difference $\Delta\kappa_i$ at the control points. The G^2 continuity is achieved ifwhen the tolerance is satisfied (e.g., $\Delta\kappa_i < 10^{-6}$). If it is not satisfied, find the θ_{ei} that satisfies the condition of $\kappa_{i,1} = \kappa_{i+1,0}$.
10. Use the new θ_{ei} in Step 8. to recalculate the curvature of each LA curve segment at control points until $\Delta\kappa_i$ meets the accuracy desired.

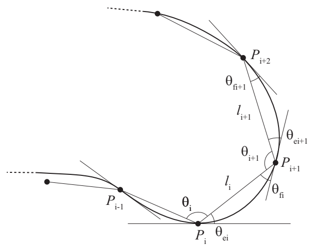

Figure 6: The notation for control points and tangential angles of LA curves.

4.1 Examples of LA Curve Generation

As shown in Fig.7, we input 9 points sampled from an example in Yan et al. [10] and form a closed control polygon with $\alpha = 0.5$. At first the algorithm generates κ-curves interpolating the control points as shown on the left with magnitude of signed curvatures in Fig.7. Then we extract tangent vectors at the control points and use this information to set the tangential angle as the initial direction angles of each LA curve segment. The middle figure shows LA curves generated using the initial direction angles. As shown in the figure G^2 continuity at the joints of the LA curves is not satisfied. For the normalized input control points, which is bounded by a unit square, after iterating about 50 to 100 times, the difference of the curvature $\Delta\kappa$ satisfies the precision of 10^{-6}. It takes several seconds to ten and more seconds using a PC with Intel Core i7-6700 at 3.40 GHz to achieve this precision. We can visually detect there is some difference of their shapes between those curves. After adjusting the direction angles, the LA curves become G^2-continuous everywhere, comparatively smoother than κ-curves. The most significant difference bewteen the κ-curve and our curve is that we can guarantee G^2 continuity even at inflection points. Similar to the κ-curve, some control point might be not at a critical point of curvature, but all local maxima of curvature magnitude appear at control points.

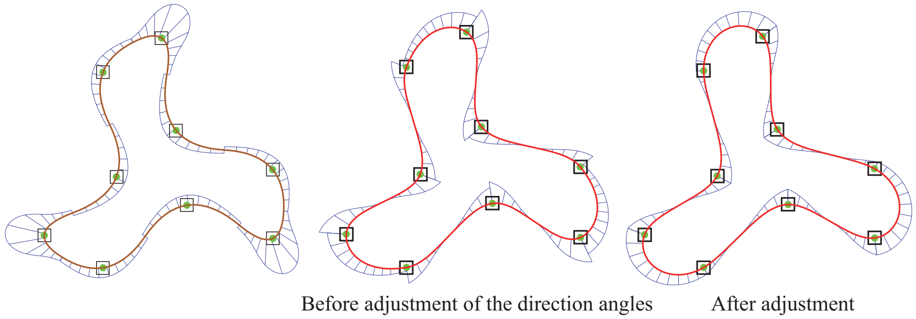

Figure 7: The comparison between κ-curves and LACs with $\alpha = -0.5$

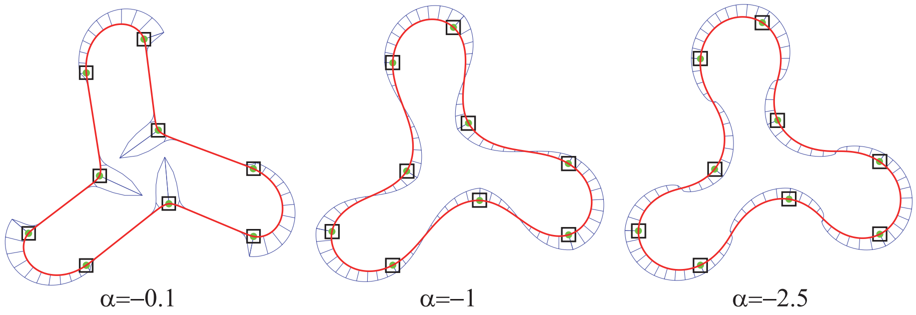

Figure 8: LA curves with various α values.

Figure 8 shows various closed LACs with different α values. When $\alpha < 0$ and in a small magnitude, i.e. -0.1, the curves are a combination of straight line and circular arc-like parts satisfying G^2 continuity everywhere. By increasing its magnitude from -0.1 to -1 to -2.5, the curves gradually become more rounded. We can accomplish subtle deformations by changing α values for fixed control points.

Figure 9 shows various LA curves with different α values. The beak tip of the bird is a start and end points of the curve and they are located at the same position satisfying G^0 continuity. It is straightforward to replace an open κ-curve with LA curves with a G^0 joint. We can use the direction angles at the start and end point of the κ-curve, which are fixed during direction angle adjustment. By changing $\alpha \in \{-0.1, -1, -2.5\}$, the curves gradually become more rounded as the in previous example.

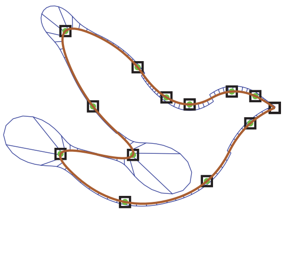

Figure 9: LA curves with its curvature magnitude drawn using various α values, with a G^0 joint at the beak.

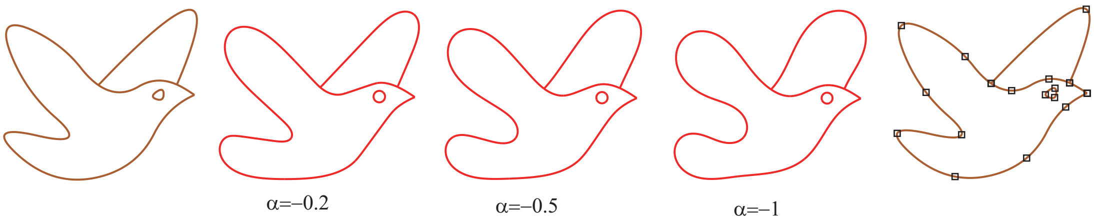

Figure 10: Different shape of bird with various α values

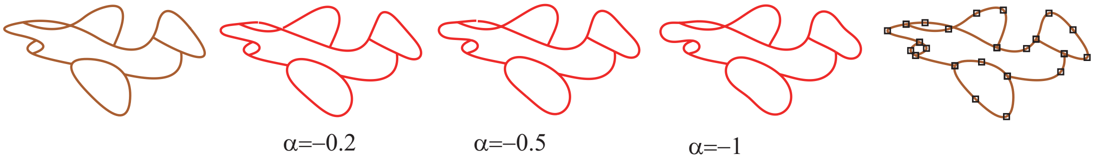

Figure 11: An aeroplane with various α values

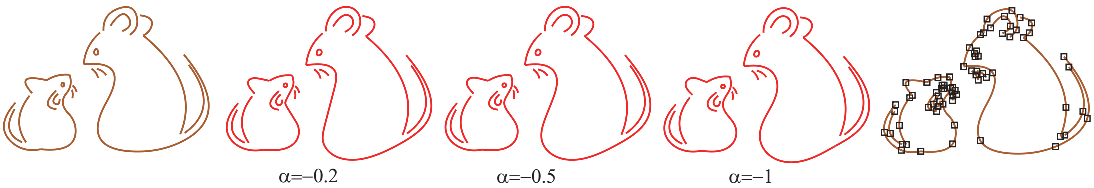

Figure 12: Mice with various α values

Figures 10, 11 and 12 show combinations of closed and open curves with various α values. In each figure, the most left is the same κ-curves generated by control points in placed in the most right in figure. The shape of the bird wings, the heads of the planes, and the bellies of the mother and child mice clearly shows that we can achieve subtle deformations by changing α values.

Figure 13 shows two examples with relatively dense control points using $\alpha = -0.5$. We obtained very similar shapes even if we change α.

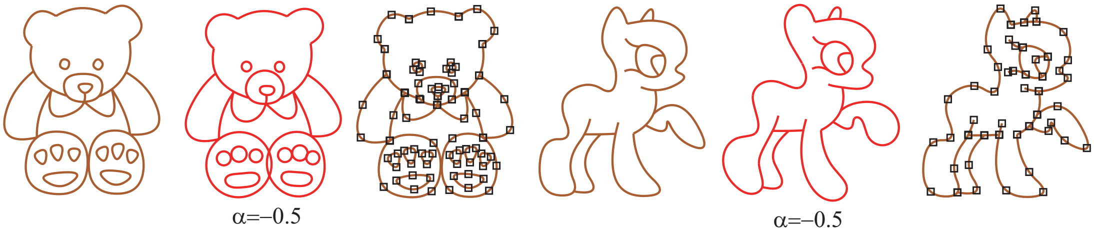

Figure 13: Bear and Bambi with various α values

4.2 Discussions

Our method generates a G^2 continuous LA curves. Hence if the number of control points is three, the algorithm generates a circle as shown in Fig. 14. The most left figure is a κ-curve with its control points. The second curve is generated using LA curves before the adjustment of direction angles and the third figure is obtained after adjustment where we obtained a perfect circle.

However if the positions of three control points are irregular, for example three points are pushed to be flat, although the initial LA curves are proper, the algorithm might be unstable and sometimes might not converge to a desired shape. To avoid this phenomenon, a possible solution is to add one more control point as shown for the bear's paws in Fig. 13.

Figure 15 shows two examples on deforming LA curves by changing the positions of input points and adding extra control points. We can make the wing thinner by moving two control points in red sharper by adding two extra control points.

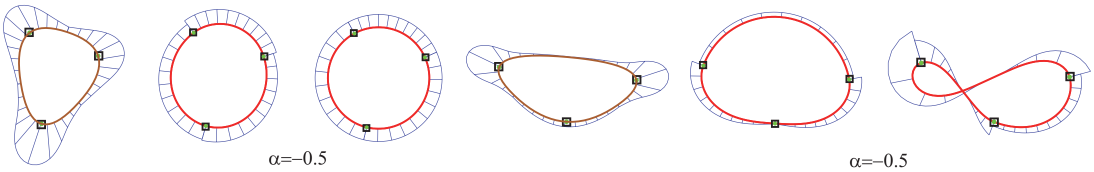

Figure 14: Circle generation and non-convergent case

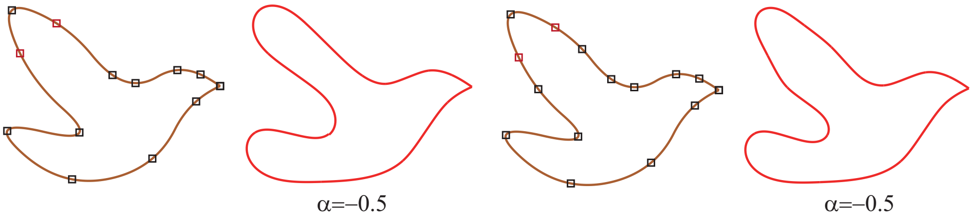

Figure 15: Deformation by changing the positions of two input points and adding extra two control points

5 CONCLUSION & FUTURE WORK

We have proposed a novel method that enables a set interpolating curves by shape parameters and guarantees G^2 continuity at every control point by replacing quadratic Bézier curve with LA curves. We have also proposed a numerical method to determine C-shaped or S-shaped LA curves clearly based on given boundary condition and α values. We have also compared the shapes between κ-curves and LA curves. By increasing the magnitude of the negative value of α in LA curves, we obtain gradually rounded shapes.

Work in progress include to speed up processing time of the proposed method using GPU and parallel programming approach. For speeding up, we might be able to use discrete log-aesthetic curves [9] to detect appropriate tangent vectors at the input points and finally generate continuous log-aesthetic curves once. An alternative approach will be to use bi-LAC proposed by Gobithaasan et al. [3], σ-curve [7] and τ-curve [8] for deformation and various CAD practicalities.

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