The Trefoil Spline

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Abstract. The Trefoil is an algebraic curve with many remarkable properties. We propose its use as the generating curve for a new family of splines.

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1 INTRODUCTION

We present a study pertaining to a family of curves defined by a variational problem arising from the Localized Induction Equation. Among the solutions to the problem, two special ones are elastic curves. The third is the Kiepert trefoil, which has a number of remarkable properties. We will relate the trefoil to the study of the fairness of splines.

Fairness is essentially the notion of smoothness. It is more of an aesthetic property than a mathematical one, since fairness depends on how humans perceive curves. We claim that our variational problem is a fairness metric for splines. The trefoil is a model of a two parameter family of splines. In this paper, we will show that it satisfies the conditions to be such a model, explore its properties and provide a few examples of its application in font design.

2 THE TREFOIL IN HISTORY

The Trefoil is defined by the equation $r^3 = a^3\cos(3\phi)$, where r and ϕ are the polar coordinates of a point in the plane. In rectangular coordinates,

$$(x^2 + y^2)^3 = a^3(x^3 - 3xy^2)$$

It is part of a family of curves known as Sinusoidal Spirals (although they are not generally spirals) given by

$$r^n = a^n\cos(n\phi),$$

where n is usually integral or at least rational. This family of curves was apparently first studied by Colin Maclaurin in 1718[11]. Most references (e.g.,[8],[16]) mention that this family includes the rectangular hyperbola ($n = -2$), the parabola ($n = -1/2$), the cardioid ($n = 1/2$), the circle ($n = 1$), and the Bernoulli Lemniscate ($n = 2$).

Zwikker[17] refers to the family as sinus spirals. He illustrates the example of the trefoil and observes that it is an equipotential line for three equally spaced point sources on the circle.

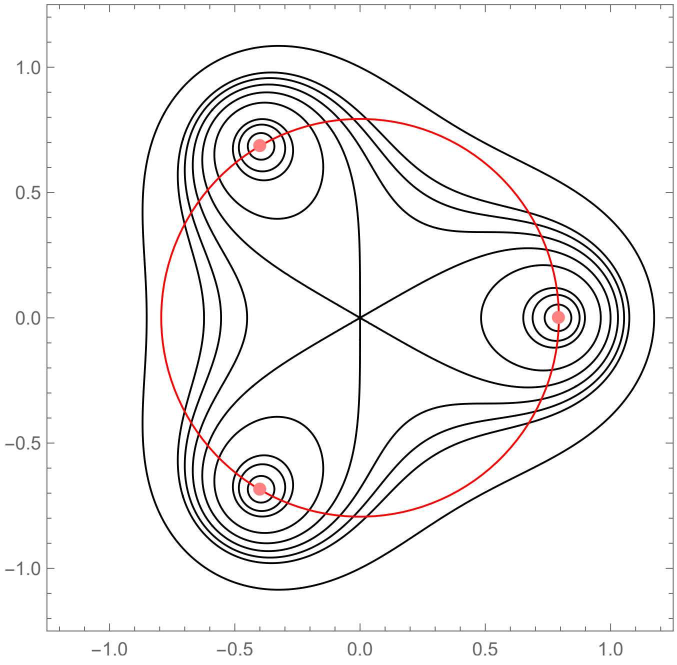

Figure 1: Equipotential lines

Cox and Shurman[4] consider the family of positive half-integers $n = \frac{m}{2}$, which they refer to as m-clovers.

The specific case of the trefoil was studied by Kiepert in his dissertation [5] in 1870. Serret had investigated the question of when an algebraic curve has arclength given by an elliptic integral of the first kind. He found a family of curves, later expanded by Liouville, specified by two parameters m and n (see [13].) Kiepert discovered curves that were not included in the family; among them was the trefoil. Shikin [15] refers to the curve as the Curve of Kiepert.

For more information and interesting properties of the trefoil, see [7].

3 THE TREFOIL AS SOLITON

The *filament equation* (also called the *Betchov-Da Rios equation* or the *localized induction equation* LIE) is an evolution equation on arc-length parameterized curves $\Gamma(s)$ in $\mathbb{R}^3$ defined as

$$\Gamma' = \dot{\Gamma} \times \ddot{\Gamma},$$

where Γ' is a derivative with respect to time and $\dot{\Gamma}, \ddot{\Gamma}$ are derivatives with respect to arc-length s along the curve Γ. It is a solition equation for space curves, best known as a model for the behavior of thin vortex tubes in an incompressible, inviscid, three dimensional fluid. This is an infinitely dimensional, completely integrable Hamiltonian system ([1] [6]). The filament equation possesses infinitely many conserved quantities, all involving integrals in terms of the curvature k and the torsion τ of a curve. This forms a hierarchy of Poisson commuting integrals which start with

$$\int 1\ ds, \quad \int \tau\ ds, \quad \int k^2\ ds, \quad \int k^2\tau\ ds, \quad \int \left(\dot{k}^2 + k^2\tau^2 - \frac{1}{4}k^4\right) ds, ...$$

Each of these integrals defines a variational problem for curves $\gamma(s)$; the resulting equilibria provide initial conditions $y(s,0) = \gamma(s)$ for soliton solutions to LIE. From $\int 1\ ds$, the equilibria are geodesics. The equilibria for the linear combinations of $\int 1\ ds$ and $\int \tau\ ds$ form helices. The equilibria for the linear combinations of $\int 1\ ds$, $\int \tau\ ds$ and $\int k^2\ ds$ form Kirchhoff elastic rods ([6]). For the purpose of this paper, we will deal with plane curves in $\mathbb{R}^2$, where the torsion element $\tau = 0$. With this in mind, we analyze the next integral in line that does not completely vanish when $\tau = 0$, $\int \left(\dot{k}^2 - \frac{1}{4}k^4\right) ds$.

Using the standard techniques, we derive the variation formulas for k^2, k^4, $\dot{k}^2$. If W is a variation vector field along a curve γ, with T the unit tangent field, then we have

$$\begin{gathered} W(v) = -gv, g = [W,T] = -\langle \nabla_T W, T\rangle \\ W(k^2) = 2\langle \nabla_T\nabla_T W, kN\rangle + 4gk^2 \\ W(k^4) = \langle \nabla_T\nabla_T W, 4k^3N\rangle + 8gk^4 \\ W(\dot{k}^2) = 2\langle \nabla_T\nabla_T\nabla_T W, \nabla_T\nabla_T T\rangle - \langle \nabla_T\nabla_T W, 4k^3N\rangle \\ -2\ddot{g}k^2 + 6\dot{g}k\dot{k} + 6g\dot{k}^2 - 2gk^4 \end{gathered}$$

Then we compute

$$\begin{gathered} \frac{d}{dw}\int \frac{1}{2}\dot{k}^2 - \frac{1}{8}k^4 ds \\ = \int \langle \nabla_T W, J\rangle ds, \quad J = (\frac{3}{8}k^4 - \frac{1}{2}\dot{k}^2 + k\ddot{k})T + (\dddot{k} + \frac{3}{2}k^2\dot{k})N \end{gathered}$$

The equilibria are then seen to be the solutions to $\nabla_T J = 0$. This gives the Euler-Lagrange equation for the curvature k:

$$E = \ddddot{k} + \frac{5}{2}k^2\ddot{k} + \frac{5}{2}k\dot{k}^2 + \frac{3}{8}k^5 = 0 \tag{3.1}$$

Now assume that a solution to Equation 3.1 has the form

$$\frac{1}{2}\dot{k}^2 = -\frac{1}{8}k^4 + \phi(k) \tag{3.2}$$

Then ϕ satisfies the third order equation

$$0 = 2\phi'''\phi - \frac{1}{4}k^4\phi''' + \phi''\phi' + k^2\phi' - \frac{1}{2}k^3\phi'' - k\phi, \qquad ' = \frac{d}{dk} \tag{3.3}$$

which has first integral

$$C = 2\phi\phi'' - \frac{1}{2}\phi'^2 - \frac{1}{4}k^4\phi'' + \frac{1}{2}k^3\phi' - \frac{1}{2}k^2\phi \tag{3.4}$$

This has polynomial solutions

$$\phi_1(k;a) = ak^2 + 4a^2 \quad (C = 16a^4) \tag{3.5}$$

and

$$\phi_2(k;a) = bk \quad (C = -\frac{1}{2}b^2) \tag{3.6}$$

Solving Equation 3.2 in the case $\phi = \phi_1$ yields the Jacobi elliptic function

$$k = A\mathrm{cn}(\alpha s, p) \quad A = 2\alpha p$$

where the elliptic modulus is

$$p^2 = \frac{3-\sqrt{3}}{6} \quad \text{or} \quad p^2 = \frac{3+\sqrt{3}}{6}.$$

The corresponding solution curves are certain elastic curves.

Solving Equation 3.2 in the case $\phi = \phi_2$ yields the Weierstrass elliptic function

$$k = \frac{1}{\wp(s; 0, \frac{1}{4})}$$

The corresponding solution curve is the Kiepert trefoil.

Remark 3.1 *We observe in passing that the trefoil is seen to be a soliton solution to LIE that evolves by rotation.*

4 A CRITERION FOR SUITABLE SPLINES

A *spline curve* in the plane is a curve passing consecutively through a specified set of points $P_1, P_2, ..., P_n$, called knots. The segment of the curve joining P_i and P_{i+1} is a smooth curve, but the entire curve is typically piecewise smooth, with at least a tangent line at each knot (See Figure 2). While mathematical descriptions of splines often assume each piece is given by polynomials, we will be interested in more general ("nonlinear") splines.

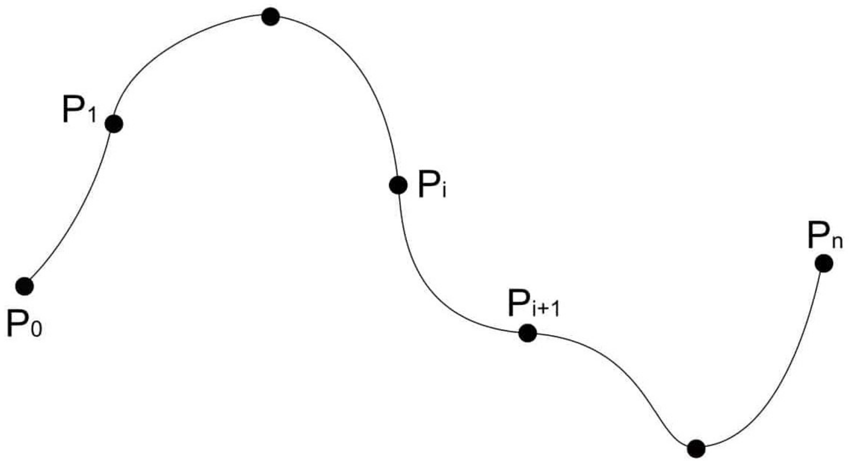

Figure 2: An example of a spline

For applications in graphic design, one goal is the construction of aesthetically pleasing curves passing through a sparse set of knots. But as of this time, there is no agreement as to what qualifies as the "best" spline. In [10], Levien and Sequin list a number of ideal properties that would be desirable for splines to have, which collectively constitute "fairness." These properties include smoothness (at least curvature-continuous), roundness, extensionality and locality, among others. They propose that for these purposes that splines should be chosen from an *extensional, two parameter* family of curves.

A two-parameter spline is a spline such that every interpolating curve segment between two adjacent knots is uniquely determined by the two angles between tangent and chord at the endpoints of the segment. In his thesis, Levien showed that there was a special relationship between two-parameter and extensional splines. He found that all two-parameter extensional splines correspond to a single generating curve, where segments between adjacent knots can be cut from such a curve (subject to scaling, rotation, translation and mirror image transformations). This is done so that the endpoints of the cut segments can be aligned with the endpoints in the spline. In addition to that, any generating curve can be used as the basis of a two-parameter extensional spline . This is stated in his theorem ([9], Theorem 1;[10]):

Theorem 4.1 (Levien) *In an extensional, G^2-continuous, two-parameter spline, there exists a curve such that for any spline segment (parametrized by angles θ_0 and θ_1) there exist two points on the curve (s_0 and s_1) such that the segment of the curve, when transformed by rotation, scaling and translation so that the endpoints coincide, also coincides along the length of the segment.*

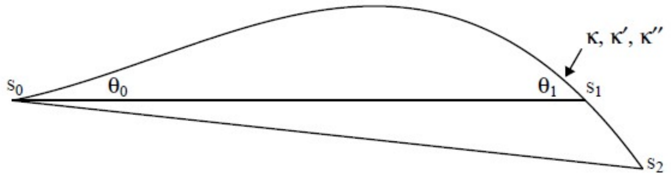

Figure 3: Construction of the generator curve of a two-parameter spline

Levien states that the proof to this theorem revolves around the quantity $\dot{k}/k^2$, which he asserts is limit of the quantity $6(\theta_1 - \theta_0)/(\theta_0 + \theta_1)^2$ as both θ_0 and θ_1 approach zero (i.e. as the length of the curve segment becomes infinitesimal). This particular quantity represents how much curvature variation there is for a segment for a fixed curvature. In the following Theorem ([3],Theorem 3.2.2), we analyze the quantity $\dot{k}/k^2$ for an arbitrary curve.

Let $X(t)$ be a smooth curve defined on an interval $[a, b]$. Assume that the curvature k satisfies the condition $\dot{k}/k^2$ is a strictly monotonic function on (a, b), where $\dot{k}$ is the derivative of k with respect to arc length.

Theorem 4.2 *Given $a < u < v < b$ and let V be the secant line from $X(u)$ to $X(v)$. Let $\theta_0 > 0$ be the angle between the tangent vector $T(v)$ at $X(v)$ and V. Likewise, let $\theta_1 > 0$ be the angle between the tangent vector $T(u)$ at $X(u)$ and V.*

Then for any sufficiently close pair of angles θ_0' and θ_1', there exists a unique nearby pair of points $X(u')$ and $X(v')$ such that the corresponding secant line achieves these angles.

Proof. We define the following function:

$$F(v, u) = (\theta_0, \theta_1) = (\arcsin\langle V, N(v)\rangle, \arcsin\langle V, N(u)\rangle).$$

We now find the Jacobian J of this multivariable function where

$$J = \begin{vmatrix} \frac{\partial}{\partial u}\theta_0 & \frac{\partial}{\partial v}\theta_0 \\ \frac{\partial}{\partial u}\theta_1 & \frac{\partial}{\partial v}\theta_1 \end{vmatrix}.$$

where

$$\begin{aligned}
\frac{\partial}{\partial u}\theta_0 &= \frac{1}{\cos\theta_0}\Big(-\frac{1}{r^3}\langle T(u), X_{10}\rangle\langle X_{10}, N(v)\rangle + \frac{1}{r}\langle T(u), N(v)\rangle\Big) \\
\frac{\partial}{\partial v}\theta_0 &= \frac{1}{\cos\theta_0}\Big(\frac{1}{r^3}\langle T(v), X_{10}\rangle\langle X_{10}, N(v)\rangle + \frac{1}{r}\langle X_{10}, -k(v)T(v)\rangle\Big) \\
\frac{\partial}{\partial u}\theta_1 &= \frac{1}{\cos\theta_1}\Big(-\frac{1}{r^3}\langle T(u), X_{10}\rangle\langle X_{10}, N(u)\rangle + \frac{1}{r}\langle X_{10}, -k(u)T(u)\rangle\Big) \\
\frac{\partial}{\partial v}\theta_1 &= \frac{1}{\cos\theta_1}\Big(\frac{1}{r^3}\langle T(v), X_{10}\rangle\langle X_{10}, N(u)\rangle + \frac{1}{r}\langle -T(v), N(u)\rangle\Big)
\end{aligned}$$

where

$$\frac{\partial}{\partial u}\Big(\frac{1}{r}\Big) = -\frac{1}{r^3}\langle T(u), X(u) - X(v)\rangle \quad \text{and} \quad \frac{\partial}{\partial v}\Big(\frac{1}{r}\Big) = \frac{1}{r^3}\langle T(v), X(u) - X(v)\rangle.$$

For the Jacobian, we wish to find in which case(s) does $J \neq 0$. We wish to show that the derivative of the function F is non-singular and is locally homeomorphic near 0. To make the Jacobian a bit easier to calculate, we will change the variables from (v, u) to $(v, v + e)$. This means we are in the half plane $e > 0$. We will also fix v at 0 and let e become s. So by the change of variables, we transform (v, u) to $(0, s)$. By using Taylor series centered at 0 and truncating all expansions to the cubic term, we arrive at the following computations:

$$\begin{aligned}
X(s) - X(0) = rV &= sT + \frac{s^2}{2}kN + \frac{s^3}{6}(\dot{k}N - k^2T) \\
T(s) &= T + skN + \frac{s^2}{2}(\dot{k}N - k^2T) + \frac{s^3}{6}((\ddot{k} - k^3)N - 3k\dot{k}T) \\
N(s) &= N - skT + \frac{s^2}{2}(-\dot{k}T - k^2N) - \frac{s^3}{6}((\ddot{k} - k^3)T + 3k\dot{k}N)
\end{aligned}$$

where

$$T = T(0), \quad N = N(0), \quad k = k(0), \quad \dot{k} = \dot{k}(0) \quad \text{and} \quad \ddot{k} = \ddot{k}(0).$$

Then, by making the necessary substitutions and dot products, we have

$$J = \frac{\frac{1}{12}(2\dot{k}^2 - k\ddot{k})s^8 + \frac{1}{24}\dot{k}(k^3 + 2\ddot{k})s^9 + \frac{1}{24}k^3\ddot{k}s^{10}}{r^6\cos\theta_0\cos\theta_1}.$$

Since the angles are small when s is small, we can use the small angle approximations for cosine which yield the product $\cos\theta_0\cos\theta_1 \approx 1$. The value r^2 is approximately s^2 when s is small, and so r^6 is approximately s^6 when s is small. Thus, we can estimate J by a polynomial of lowest order term s^2

$$J \approx \frac{1}{12}(2\dot{k}^2 - k\ddot{k})s^2 + \frac{1}{24}\dot{k}(k^3 + 2\ddot{k})s^3 + \frac{1}{24}k^3\ddot{k}s^4.$$

If $s = 0$, the Jacobian is singular. But if $s \neq 0$ but is sufficiently small, the higher order terms are irrelevant and the lowest order term $\frac{1}{12}(2\dot{k}^2 - k\ddot{k})s^2$ dominates.

The assumption that $\dot{k}/k^2$ is strictly monotonic means that

$$\frac{d}{ds}\left(\frac{\dot{k}}{k^2}\right) = \frac{k^2\ddot{k} - 2k\dot{k}^2}{k^4} = -\frac{2\dot{k}^2 - k\ddot{k}}{k^3} \neq 0.$$

Thus, $J \neq 0$. ■

Because a Taylor expansion for the curve and for the tangent and normal vectors was used, the theorem produces a local result. In particular, the map is locally invertible. This, combined with an upcoming theorem (7.3), will generate a global result.

5 THE MINIMUM ENERGY CURVE AND THE MINIMUM VARIATION CURVE

The Minimum Energy Curve (MEC) is the mathematical idealization of a flexible strip that is constrained to go through all the control points, so that the strip can slide freely. This curve minimizes the bending energy. Mathematically, the MEC is the curve which minimizes the following functional:

$$E_{MEC} = \int_0^l k^2 \, ds.$$

Lee and Forsythe, who both did extensive work with the MEC, showed that the spline can be defined as piecewise segments of the rectangular elastica between each pair of adjacent control points, with G^2 continuity across them. It was proposed that the bending energy would make a good fairness metric. While it is the optimum of the bending energy, this does not make the best spline. The MEC is a two-parameter, extensional spline. So, the shape of the curve above any given chord can be completely determined by the two tangent directions of the curve at either end of the chord. Despite having such desirable properties, it lacks in roundness and robustness. But most of all, it does not accurately predict perceived smoothness. This key result was discovered by Raph Levien in his thesis, where he conducted a survey where he asked participants to pick from a family of curves to determine which one was more fair to them. He found that the concept of fairness is fuzzy at best and varies based on who is being asked [9]. The MEC also fails to be scale invariant, meaning that zooming in and out and redoing the calculation does not produce the same curve [2].

The Minimum Variation Curve (MVC) is defined as the curve minimizing the integral of square of the arc-length derivative of curvature,

$$\int_0^l \dot{k}^2 \, ds.$$

Moreton, who introduced this in 1992, argued that the four-parameter MVC spline is a better alternative to the two-parameter MEC spline due to its limitations. For one, the MEC fails the roundness property. But because the MVC forms circular arcs naturally, a circular arc has zero curvature variation. Trivially, it is the curve that minimizes the MVC cost functional when the input points are co-circular [10], [12]. Second, the MVC is extensional, along with obtaining locality. Third, while the MEC tries to find the curve that bends the least, the MVC bends as smoothly (uniformly) as possible and is more stable to changes in shape [2]. Fourth, it is naturally convex preserving since it is guaranteed not to have any extraneous points of inflection [12]. Despite these advantages that the MVC has over the MEC, fairness is not captured by the minimization of the variation of curvature either.

Just like the MEC functional, the MVC functional can be explored via variational formulas. By using similar techniques, we arrive at the Euler-Lagrange equation:

$$E = \ddddot{k} + k^2\ddot{k} + \frac{1}{2}k\dot{k}^2 = 0.$$

Since fourth-order nonlinear differential equations are difficult (and perhaps impossible) to solve analytically, most presentations of this curve use a numerical approach to minimize the cost functional.

6 A COMMENT ON ABERRANCY

Recall from Theorem 4.2 that a more geometric proof was presented as to how and why the quantity $\dot{k}/k^2$ is key to Levien's proof. The quantity $\dot{k}/k^2$ actually has a geometric interpretation that has been known since the 1800s but has received little to no attention. This quantity is called the *déviation* of a curve, or is better known as the *aberrancy* of a curve. Geometrically, the aberrancy of a curve $f(x)$ at a point P is the tangent of the angle δ formed between the normal at P and the limiting position of a line drawn from P to the midpoint of a chord parallel to the tangent line at P as the chord approaches P ([14]).

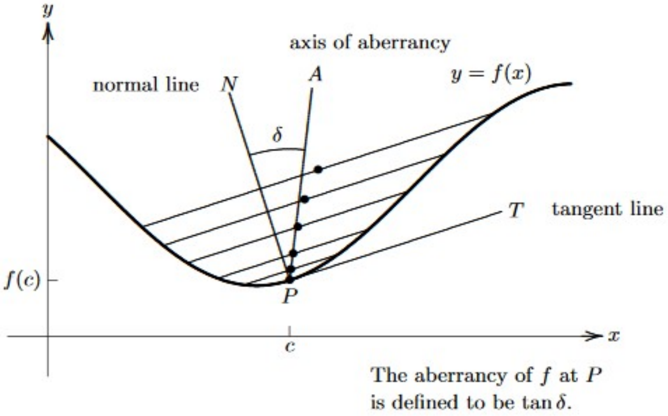

Figure 4: The aberrancy of a curve $f(x)$ at point P

For the trefoil, its curvature satisfies the condition $\dot{k}^2 + \frac{1}{4}k^4 = 8k$. Thus

$$\dot{k}^2 = -\frac{1}{4}k^4 + 8k.$$

By taking the derivative with respect to s on both sides, we get

$$\ddot{k} = -\frac{1}{2}k^3 + 4.$$

This implies

$$k^2\ddot{k} = -\frac{1}{2}k^5 + 4k^2 \quad \text{and} \quad 2k\dot{k}^2 = -\frac{1}{2}k^5 + 16k^2$$

which yields

$$k^2\ddot{k} - 2k\dot{k}^2 = -12k^2 < 0.$$

Thus, for the trefoil, the quantity $\dot{k}/k^2$ is monotonic.

7 THE TREFOIL AS A SUITABLE SPLINE

So why is the trefoil actually a good candidate for a spline? Recall that Theorem 4.2 gave a geometric reasoning behind Levien's claim and the aberrancy quantity $\dot{k}/k^2$ for general curves. However, the theorem is only true locally. The trefoil is actually a fascinating curve since it satisfies the conditions in the theorem globally, which is important for applications to splines. To show why the trefoil satisfies Theorem 4.2 globally, we will need two propositions.

First, the polar coordinates of the trefoil is $r^3 = 2\cos(3\phi)$. Using ϕ as a parameter, we have

$$X(\phi) = (r\cos(\phi), r\sin(\phi)) = rU \tag{7.1}$$

with $U = (\cos(\phi), \sin(\phi))$ and $V = (-\sin(\phi), \cos(\phi))$ being an orthonormal frame along the curve. By implicit differentiation, we get

$$\frac{dr}{d\phi} = -\frac{2\sin(3\phi)}{r^2} = -\frac{2r\sin(3\phi)}{2\cos(3\phi)} = -r\tan(3\phi).$$

This implies that

$$\begin{aligned} X'(\phi) = rV - r\tan(3\phi)U &= \frac{r}{\cos(3\phi)}(\cos(3\phi)V - \sin(3\phi)U) \\ &= \frac{r}{\cos(3\phi)}T, \end{aligned} \tag{7.2}$$

where

$$\begin{aligned} T &= (-\cos(3\phi)\sin(\phi) - \sin(3\phi)\cos(\phi), \cos(3\phi)\cos(\phi) - \sin(3\phi)\sin(\phi)) \\ &= (-\sin(4\phi), \cos(4\phi)). \end{aligned}$$

With this preliminary, we have our first proposition:

Proposition 7.1 *For the trefoil, the following formula holds:*

$$\theta_0 + \theta_1 = 4(\phi_0 - \phi_1),$$

where θ_0, θ_1 are the secant angles at two distinct points and ϕ_0, ϕ_1 are the polar angles of those corresponding points.

To see why this is true, suppose $P = X(\phi_0)$ and $Q = X(\phi_1)$ are two points on the leaf of the trefoil. Consider the triangle ABC formed by the x-axis and the tangent lines to the curve at P and Q. Call these tangent lines $T_0 = (-\sin(4\phi_0), \cos(4\phi_0))$ and $T_1 = (-\sin(4\phi_1), \cos(4\phi_1))$, respectively. Let ψ be the vertex angle at C, where the two tangent lines intersect. Let θ_0 and θ_1 denote the angles between the secant line PQ and the tangents at P and Q. Then $\psi = \pi - \theta_0 - \theta_1$. The base angles of the triangle at A and B can be determined to be $\measuredangle A = 4\phi_0 - \frac{\pi}{2}$ and $\measuredangle B = \frac{\pi}{2} - 4\phi_1$. Comparing, the following formula holds:

$$\theta_0 + \theta_1 = 4(\phi_0 - \phi_1).$$

This formula is remarkable since it gives us a relationship between the unknown polar angles ϕ_0, ϕ_1 and the known angles between the secant lines θ_0, θ_1. The next logical step would be to find another relationship between these angles to solve for the polar angles, given any two secant angles. This relationship gives us our second proposition:

Proposition 7.2 *For the trefoil, the following formula holds:*

$$\cos^3(3\phi_0 - \theta_0)\cos(3\phi_0) = \cos^3\left(3\phi_0 + \frac{1}{4}(\theta_1 - 3\theta_0)\right)\cos(3\phi_1),$$

where θ_0, θ_1 are the secant angles at two distinct points and ϕ_0, ϕ_1 are the polar angles of those corresponding points.

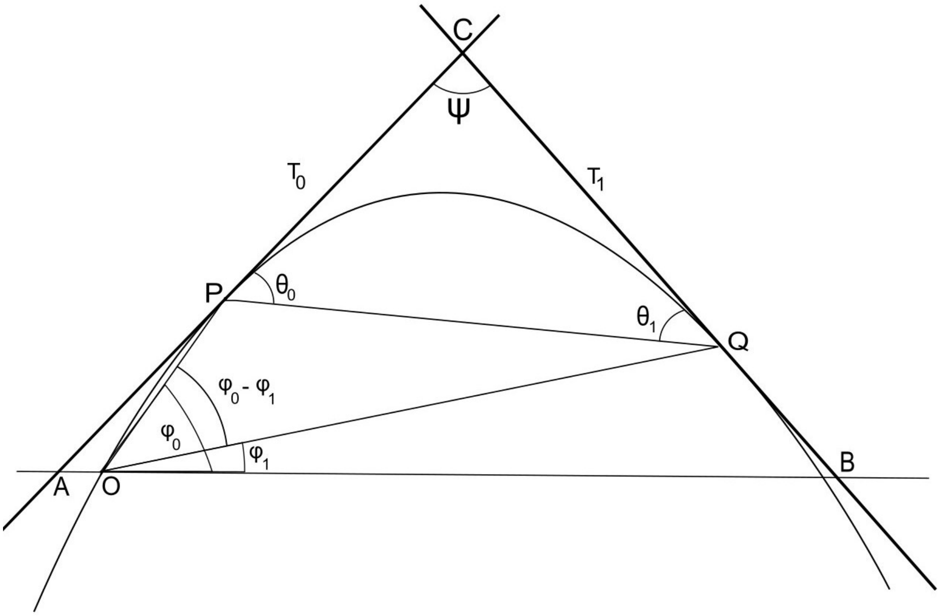

Figure 5: Tangents to the trefoil leaf

Proof. Let $A = \phi_0 - \phi_1$. In triangle OPQ of Figure 5, we have $|OP|^3 = 2\cos(3\phi_0$ and $|OQ|^3 = 2\cos(3\phi_1)$. $\angle OPQ = \frac{\pi}{2} + 3\phi_0 - \theta_0$ and

$$\angle OQP = \frac{\pi}{2} - 3\phi_1 - \theta_1 = \frac{\pi}{2} - 3\phi_0 + 3A - (4A - \theta_0)$$

$$= \frac{\pi}{2} - 3\phi_0 + \theta_0 - A$$

So $\sin(\angle OPQ) = \cos(3\phi_0 - \theta_0)$ and $\sin(\angle OQP) = \cos(3\phi_0 + A - \theta_0)$. ■

As we can see, the proposition follows from an argument using trigonometry with information from Figure 5. With these two propositions, we have enough to state the theorem which proves that our trefoil satisfies the global result of Theorem 4.2.

Theorem 7.3

For the trefoil leaf, there is a one-to-one correspondence between pairs of distinct points and pairs of angles θ_0 and θ_1 satisfying the constraints: $1.)\quad 0 < \theta_0 + \theta_1 < \dfrac{4\pi}{3}$ *and* $2.)\quad \dfrac{1}{3} \leq \dfrac{\theta_1}{\theta_0} \leq 3.$

Proof. A pair of distinct points $P = X(\phi_0)$ and $Q = X(\phi_1)$ on a leaf of the trefoil is specified by a pair of angles (ϕ_0, ϕ_1) satisfying $-\dfrac{\pi}{6} < \phi_1 < \phi_0 < \dfrac{\pi}{6}$ and a corresponding pair of angles (θ_1, θ_2) satisfying $0 < \theta_0 + \theta_1 < \dfrac{4\pi}{3}$, as seen in figure 5. Note that for fixed Q, the maximum value of θ_0 occurs when P is at the origin O and $\phi_0 = \dfrac{\pi}{6}$; it decreases to 0 as P approaches Q. Likewise, for fixed P, θ_0 is maximized for $Q = O$; that is, when $\phi_1 = -\dfrac{\pi}{6}$.

When $\phi_0 = \dfrac{\pi}{6}$, the secant line from P to Q is just the radial line from the origin to Q. In that case, $\theta_0 = \frac{\pi}{6} - \phi_1$, while $\theta_1 = \frac{\pi}{2} - 3\phi_1$. The ratio $\frac{\theta_1}{\theta_0} = 3$. Similarly, when $\phi_1 = -\dfrac{\pi}{6}$, the ratio $\frac{\theta_1}{\theta_0} = \frac{1}{3}$.

If we vary the positions of P and Q, beginning at $\phi_0 = \frac{\pi}{6}$ and $Q = \phi_1 = \frac{\pi}{6} - \triangle$, and keeping $\phi_0 - \phi_1 = \triangle$ constant, then $\theta_0 + \theta_1 = 4\triangle$ will be constant. By continuity, the ratio $\frac{\theta_1}{\theta_0}$ must take on every value from 3 to $\frac{1}{3}$.

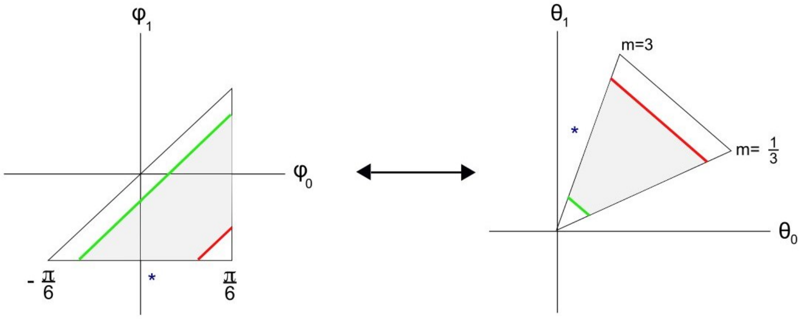

Figure 6: Pictorial reasoning behind Theorem 7.3

We have a map from the triangular region illustrated on the left of Figure 6 to the triangular region on the right. Each line segment of slope 1 in the ϕ-plane is mapped to a line of slope -1 in the θ-plane. By the argument above, we see that the image of the map contains the entire triangle bounded by the rays of slope 3 and slope $\frac{1}{3}$ and the line $\theta_0 + \theta_1 = \frac{4\pi}{3}$. ■

8 NUMERICAL COMPUTATION OF TREFOIL SPLINES

In this section, we observe the fact that the trefoil satisfies Theorem 4.2 globally based on Theorem 7.3. Given angles θ_1 and θ_2, we will determine the corresponding angles ϕ_1 and ϕ_2. This gives us the angle coordinates of the two points P and Q of the trefoil, from which the resulting arc can then be explicitly generated by using the polar coordinate formula for the trefoil. It is worth noting that solving for a specific choice of angles θ_0, θ_1 can be done numerically, which miraculously involves no difficult computations!

Recall from the previous section that we saw the relation:

$$\phi_0 - \phi_1 = \frac{1}{4}(\theta_0 + \theta_1)$$

Given θ_0, θ_1 and a polar angle ϕ_0, we can find the unique corresponding angle ϕ_1.

From equation 7.1 $P = X(\phi_0)$ and $Q = X(\phi_1)$. Let $V = \dfrac{P-Q}{\|P-Q\|}$, where V is a unit vector. The tangent vector at P is $T(\phi_0) = (-\sin(4\phi_0), \cos(4\phi_0))$. Then θ_0 is the angle between V and $-T$. The complementary angle $\pi/2 - \theta_0$ is the angle between V and the inward normal $N = (-\cos(4\phi), -\sin(4\phi))$. So

$$f(\phi, \triangle) = N \cdot V = \cos(\pi/2 - \theta_0) = \sin(\theta_0)$$

Given ϕ_0, and $\triangle = \phi_0 - \phi_1$, we can compute the terms P, $Q = X(\phi_0 - \triangle)$,$X_{10}$, $\|X_{10}\|$, and V. Thus we can compute $N \cdot V = \sin\theta_0$. By taking the arcsine of both sides, we get an expression for θ_0.

Likewise, the inverse to this can be found numerically: Given $\triangle = \frac{1}{4}\theta_0 + \theta_1$ and the measure of angle θ_0, we can find the unique measure of ϕ_0. This is done by solving the equation

$$\arcsin(f(\phi, \triangle)) + \theta_0 = 0.$$

But this is actually very easy to do since we know θ_0 and $\triangle$, because this reduces to a problem of root finding, using either Newton's method or the Bisection method.

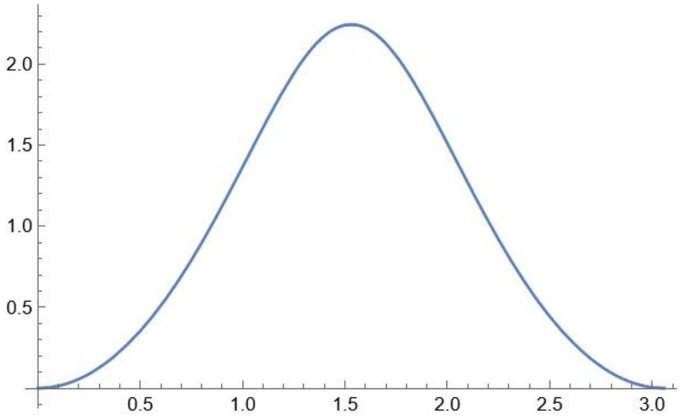

Figure 7: The curvature plot of one leaf of the trefoil over one period.

9 EXAMPLES OF THE NUMERICAL COMPUTATION

The good news about this numerical computation is that it provides a practical result opposed to just a theoretical one! A practical use for splines is in the development of fonts for computer word processors. This makes sense since the basis of all fonts are lines and circles, with required smoothing and adjustments for the specifics of certain letters, numerals and symbols.

One example would be the S-spline on the far left, which has three knots above the inflection, three knots below and one at the inflection. When we compare this with the circle spline in the center, we see that our trefoil spline on the far right actually forms the letter "S" surprisingly well.

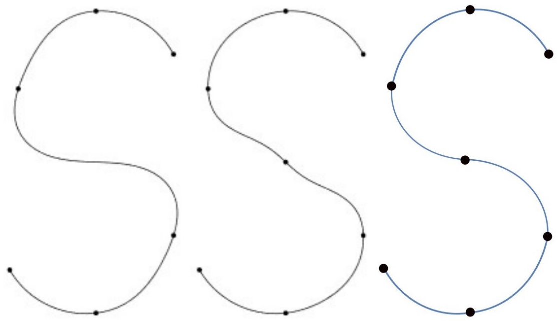

Figure 8: The S-spline formed by the trefoil, compared to the S-spline circle spline

In addition to letters, we can use pieces of the trefoil to form symbols. For instance, by using 10 knots and appropriate slices of the trefoil, we can form an ampersand which closely resembles the one on the left.

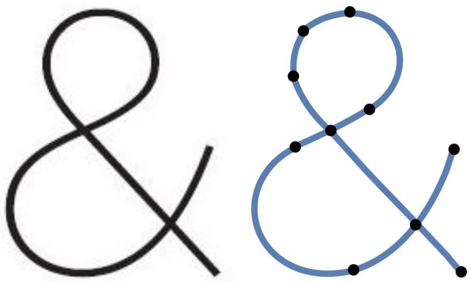

Figure 9: An ampersand formed by the trefoil spline

With 15 knots, we can form a rough (but decent) approximation of the at symbol. Of course, with more knots, the spline will become more smooth. Regardless, even with a small number of knots, we do get a nice picture.

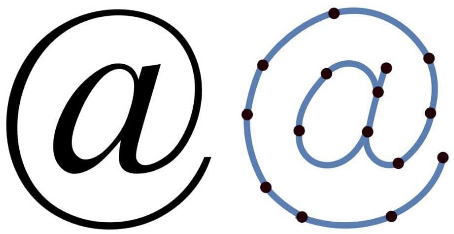

Figure 10: The at symbol formed by the trefoil spline

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10 CONCLUSIONS AND FUTURE RESEARCH

The trefoil is truly an amazing and versatile algebraic curve. Not only does the trefoil satisfy many properties but as a candidate for a spline, it is fairly accurate. In terms of the research, we only have considered planar curves in $\mathbb{R}^2$ with the filament equation. The next step would be to see what solutions come up in $\mathbb{R}^3$ if we allow the torsion element $\tau \neq 0$, i.e. $\int \left(\dot{k}^2 + k^2\tau^2 - \frac{1}{4}k^4 \right) ds$.

One topic that was not explored in this paper was the Lagrangian formulation behind the solutions to the minimization of the integral

$$\int \frac{1}{2}\dot{k}^2 - \frac{1}{8}k^4 \, ds.$$

REFERENCES

[1] Bor, G.; Levi, M.; Perline, R.; Tabachnikov, S.: Tire tracks and integrable curve evolution. International Mathematics Research Notices, 2020(9), 2698–2768, 2020. http://doi.org/10.1093/imrn/rny087.

[2] Cassel, K.W.: Variational Methods with Applications in Science and Engineering. Cambridge University Press, 2013.

[3] Clark, T.: The Trefoil: an Analysis in Curve Minimization and Spline Theory. Ph.D. thesis, Case Western Reserve University, 2020.

[4] Cox, D.; Shurman, J.: Geometry and number theory on clovers. Amer. Math. Monthly, 122, 682–704, 2005.

[5] Kiepert, F.W.A.L.: De curvis quarum arcus integralibus ellipticis primi generis exprimuntur. Dissertation, 1870.

[6] Langer, J.; Singer, D.: Liouville integrability of geometric variational problems. Comment. Math. Helvetici, 69, 272–280, 1994.

[7] Langer, J.; Singer, D.: The trefoil. Milan Journal, 82, 161–182, 2014.

[8] Lawrence, J.D.: A Catalog of Special Plane Curves. Dover Publications, Inc., 2013.

[9] Levien, R.: From spiral to spline; optimal techniques for interactive curve design. Ph.D. thesis, UC Berkeley, 2009.

[10] Levien, R.; Sequin, C.: Interpolating splines: Which is the fairest of them all? Computer Aided Design & Applications, 6, 91–102, 2009. http://doi.org/10.3722/cadaps.2009.91-102.

[11] Maclaurin, C.: Tractatus de curvarum constructione et mensura. Philosophical Transactions, 30, 803–812, 1718.

[12] Moreton, H.; Séquin, C.: Minimum variation curves and surfaces for computer-aided geometric design. In Designing Fair Curves and Surfaces: Shape Quality in Geometric Modeling and Computer-Aided Design, 123–159. Society for Industrial and Applied Mathematics, 1994. http://doi.org/10.1137/1.9781611971521.

[13] Prasolov, V.; Solovyev, Y.: Elliptic Functions and Elliptic Integrals. American Mathematical Society, 1997. http://doi.org/10.1201/b11644-8.

[14] Schot, S.: Aberrancy: Geometry of the third derivative. Mathematics Magazine, 51(5), 259–275, 1978.

[15] Shikin, E.: Handbook and Atlas of Curves. CRC Press, 1995.

[16] Yates, R.: Curves and Their Properties. National Council of Teachers of Mathematics, 1974.

[17] Zwikker, C.: The Advanced Geometry of Plane Curves and their Applications. Dover Publications, Inc., 1963.