Rational Generalized Trigonometric Curve: Rationalization of Generalized Trigonometric Curve

Kenjiro T. Miura[1], R.U. Gobithaasan[2], Md Yushalify Misro[3], Tadatoshi Sekine[4], Shin Usuki[5]

[1]Shizuoka University, miura.kenjiro@shizuoka.ac.jp
[2]Universiti Malaysia Terengganu, gr@umt.edu.my
[3]Universiti Sains Malaysia and Shizuoka University, yushalify@usm.my
[4]Shizuoka University, sekine.tadatoshi@shizuoka.ac.jp
[5]Shizuoka University, usuki@shizuoka.ac.jp

Corresponding author: Kenjiro T. Miura, miura.kenjiro@shizuoka.ac.jp

Abstract. κ-curves, which control one curvature extremum on each curve segment instead of the end points, are defined as a sequence of the quadratic Bézier curve with three control points. The authors have proposed generalized trigonometric basis functions consisting of $(\sin t, \cos t, 1)$ and defined the generalized trigonometric curve in order to extend κ-curves. In this study, we will show that the linear generalized trigonometric curve defined by three control points generates an elliptical arc, but cannot generate an arbitrary elliptical curve. Hence we will rationalize it to express an arbitrary elliptical arc as well as arbitrary arcs of parabola and hyperbola.

Keywords: generalized trigonometric curve, circular arcs, elliptical arcs, uniqueness theorem
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1 INTRODUCTION

On the extensions of the cubic Bézier curve with four control points, to connect multiple segments with required continuity has been strongly intended and for example, tangent and curvature continuity at the start and end points are guaranteed independently by adding extra shape parameters. Contrary to this research trend, κ-curves, which control one curvature extremum on each curve segment instead of the end points, are defined as a sequence of the quadratic Bézier curve with three control points. The authors have proposed generalized trigonometric basis functions consisting of $(\sin t, \cos t, 1)$ and defined the generalized trigonometric curve in order to extend κ-curves [6]. In this study, we will show that the linear generalized trigonometric curve defined by three control points generates an elliptical arc, but cannot generate an arbitrary elliptical curve. Hence we will rationalize it to express an arbitrary elliptical arc as well as arbitrary arcs of parabola and hyperbola.

2 κ-CURVE

The κ-curve, proposed recently by [10], is an interpolating spline which is curvature-continuous almost everywhere and passes through input points at the local curvature extrema. It has been implemented as the curvature tool in Adobe Illustrator® and Photoshop® and is accepted as a favoured curve design tool by many designers (see e.g. [2, 3]).

We consider the reasons for the success of κ-curve to be:

1. Information along contours is concentrated at local maxima of curvature.
2. Curves of low degree have smooth distribution of curvature.
3. G^2-continuous curves tend to look fairer than only G^1-continuous ones.

Attneave [1] suggested, based on his empirical study, that information along contours is concentrated in regions of high magnitude of curvature, as opposed to being distributed uniformly along the contour, and it is further concentrated at local maxima of curvature (see also [11]). Although Attneave never published the details of his methods, [7] conducted a similar experiment, and obtained the same results. Levien and Séquin [5] argue similarly, and assert that points of maximal curvature are salient features.

The curvature of a polynomial curve is given by a relatively complicated rational function [4], and its distribution might not be globally smooth. However, if the curve is of a low degree, the curvature distribution is more uniform and the curve is fairer, thus more suitable for illustration. The quadratic polynomial curve has the nice property that its curvature has only one local maximum, and its location is easily computable [10], which makes the handling of curvature extrema much easier.

Graphic designers often accept G^1 continuity as good enough for illustration. However, discontinuity remains; for example, if you join a straight line and a circular arc with G^1 continuity, the rhythm of the curve will be broken at the joint. For this reason we give pereference to G^2-continuous curves.

Figure 1 shows examples of the $\epsilon\kappa$-curve (Miura et al., 2021), which is an extension of the κ-curve and can control its curvature extrema. The left figures use cubic Bézier curves for their curve segments and the right figures use linear generalized trigonometric curves for them, which are rationalized in this paper. In these figures a is a parameter controling curvature extrema.

3 GENERALIZED TRIGONOMETRIC CURVE

In this section, we describe *the generalized trigonometric curve*. The blending functios of the curve is based on the trigonometric cubic Bernstein-like basis [9], which we are going to review first.

The trigonometric cubic Bernstein-like basis functions have an extra shape parameter α, and are defined by

$$\begin{aligned}
f_0 &= \alpha S^2 - \alpha S + C^2 = 1 + (\alpha - 1)S^2 - \alpha S,\\
f_1 &= \alpha S(1 - S),\\
f_2 &= \alpha(S^2 + C - 1) = \alpha C(1 - C),\\
f_3 &= (1 - \alpha)S^2 - \alpha C + \alpha = 1 + (\alpha - 1)C^2 - \alpha C,
\end{aligned} \tag{1}$$

where $S = \sin\frac{\pi t}{2}$, $C = \cos\frac{\pi t}{2}$, for $\alpha \in (0, 2)$, $t \in [0, 1]$. Note that these functions satisfy partition of unity, i.e., $\sum_{i=0}^{3} f_i(t) = 1$ for any α. When $\alpha = 1$, the above functions are simplified to

$$\begin{aligned}
f_0 &= 1 - S,\\
f_1 &= S(1 - S),\\
f_2 &= C(1 - C),\\
f_3 &= 1 - C.
\end{aligned} \tag{2}$$

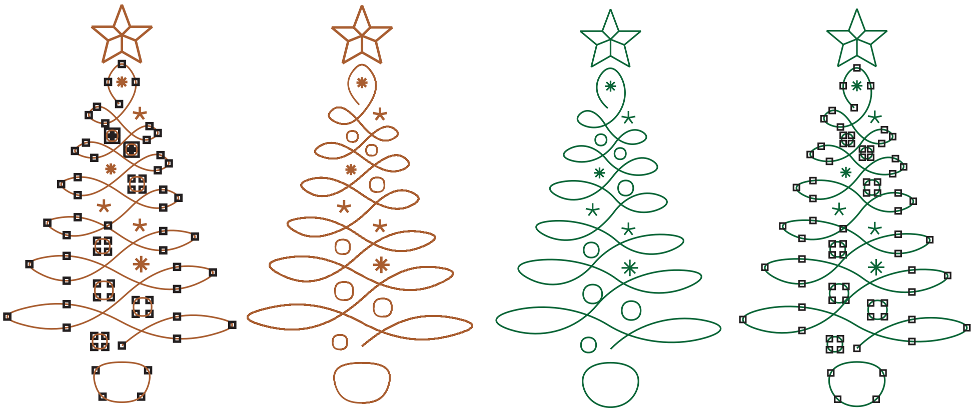


Figure 1: A Christmas tree drawn with $\epsilon\kappa$-curves using the cubic Bernstein basis functions (left) and the quadratic trigonometric basis functions (right). Mark □ indicates an input point. Note that the latter has more rounded forms and-in this case-preferable.

If we add the second and third functions together and rename them to u, v and w, we obtain blending functions $\{u, v, w\}$ as follows:

$$\begin{aligned} u &= 1 - S, \\ v &= S(1-S) + C(1-C) = S + C - 1, \\ w &= 1 - C. \end{aligned} \tag{3}$$

It is straightforward to define a curve by these blending functions with three control points, which we can regard as a "linear" generalized trigonometric curve since the highest degree the trigonometric functions are in is one.

One interesting relationship among these functions is

$$v^2 = 2uw, \tag{4}$$

which enables

$$(u + v + w)^2 = u^2 + 2uv + 4uw + 2vw + w^2, \tag{5}$$

and yields the five blending functions $\{u^2,\ 2uv,\ 4uw,\ 2vw,\ w^2\}$, associated with five control points. We can define a curve using these blending functions and regard it as a "quadratic" trigonometric curve since the highest power of each blending function is now degree two.

In a similar way, we can extend blending functions of "degree" n with $2n + 1$ control points. We can perform a recursive procedure called Gobithaasan-Miura's algorithm to evaluate a curve of any degree similar to de Casteljau's algorithm avoiding the overhead of trigonometric function evaluation. This means that it is not necessary to calculate the coefficients of blending functions, or keep a coefficient table.

In order to analyze what kind of curve can be generated by a linear generalized trigonometric curve, without loss of generality up to similarity, we specify its three control points as $(-1, 0)$, (b, h), and $(1, 0)$. When $h = 0$, the curve becomes a line segment on the x-axis and we assume that $h \neq 0$. Then the linear generalized trigonometric curve is given by

$$x = (b+1)S + (b-1)C - b \tag{6}$$

$$y = h(S + C - 1) \tag{7}$$

By using the above equations and $S^2 + C^2 = 1$ and eliminating S and C, the following equation is obtained:

$$h^2x^2 + (b^2+1)y^2 - 2b\,h\,xy + 2hy - h^2 = 0 \tag{8}$$

In the above euqation, the coefficients of x^2 and y^2 are $h^2 > 0$ and $b^2 + 1 > 0$, respectively and this equation represents an ellipse [8]. Hence, the linear generalized trigonometric curve represents an elliptical arc cut out from the ellipse. Because of symmetry of the circle, if the lengths of the two line segments connecting the control points are different, no circular arc is represented. Furthermore if we assume that the locations of the control points are made to be symmetrical along the y-axis by $b = 0$,

$$h^2x^2 + y^2 + 2hy - h^2 = \frac{1}{h^2}\left(x^2 + \frac{1}{h^2}(y^2+h)^2 - 2\right) = 0 \tag{9}$$

This equation does not represent a circle except for $h = \pm 1$ as explained below. When $h = \pm 1$, the two line segments connecting the control points become the same length and orthogonal each other and the linear generalized trigonometric curve represents a quater circular arc. Therefore in order to express an arbinrary circular or elliptical arc, its rationalization is necessary.

Although the left side of equation (8) includes the term of y and a constant, we can elliminate them by translating the curve along the y-axis. Hence it is enough to analyze the following quadratic form:

$$h^2x^2 - 2b\,h\,xy + (b^2+1)y^2 = (x, y)M\begin{pmatrix} x \\ y \end{pmatrix} \tag{10}$$

where

$$M = \begin{pmatrix} h^2 & -bh \\ -bh & (b^2+1) \end{pmatrix} \tag{11}$$

The eigenvalues λ_0, $\lambda_1$of matrix M are given by

$$\lambda_0 = \frac{1}{2}\left(b^2 + h^2 + 1 - \sqrt{b^2(b^2 + 2(h^2+1)) + (h^2-1)^2}\right) \tag{12}$$

$$\lambda_1 = \frac{1}{2}\left(b^2 + h^2 + 1 + \sqrt{b^2(b^2 + 2(h^2+1)) + (h^2-1)^2}\right) \tag{13}$$

Hence by applying an appropriate transformation, we obtain

$$\lambda_0 x^2 + \lambda_1 y^2 = r^2. \tag{14}$$

If $h \neq 0$, $\lambda_0 > 0$ and $\lambda_1 > 0$. Then the above equation represents an ellipse. Especially when $\lambda_0 = \lambda_1$, or since $b^2 + h^2 + 1 - \sqrt{b^2(b^2 + 2(h^2+1)) + (h^2-1)^2} = 0$, $b = 0$ and $h = \pm 1$. So this represents a circle. In this case, the linear generalized trigonometric curve becomes a quater circular arc. Even though we assume $b = 0$ and locate the control points symmetrically, some specific circular arc is represented and no arbitray circular arc is obtained. In the ellipse case, the number of the parameters of the implicit function expressing an ellipse is essentially 5 and one degree of freedom remains by specifying the postions of the end points and tangent vectors there (4 constraints). However the ratio of λ_0 and λ_1 is constrained as the circle, we cannot represent an arbitray circular and elliptical arc and we need its rationalization.

Figure 2 shows examples of the linear generalized trigonometric curve. To clarify its properties, we draw quadratic Bézier curves defined by the same control points at the same time. In Fig.2(a), the locations of the control points are $(0,0)$, $(1,1)$ and $(1,0)$. In (b) and (c), only the first control points are translated to $(0,1)$ and $(0,2)$. The generalized trigonomtric curves are drawn in blue and the quadratic Bézier curves in orange. From these figures, the generalized trigonometric curve has smaller absolute curvature and are more rounded than the Bézier curve. Especially in (b), the two line segments connecting the control points become the same length and orthogonal each other and its equation can be simplified as $(\sin\frac{\pi}{2}t, \cos\frac{\pi}{2}t)$. It's a quarter circular arc.

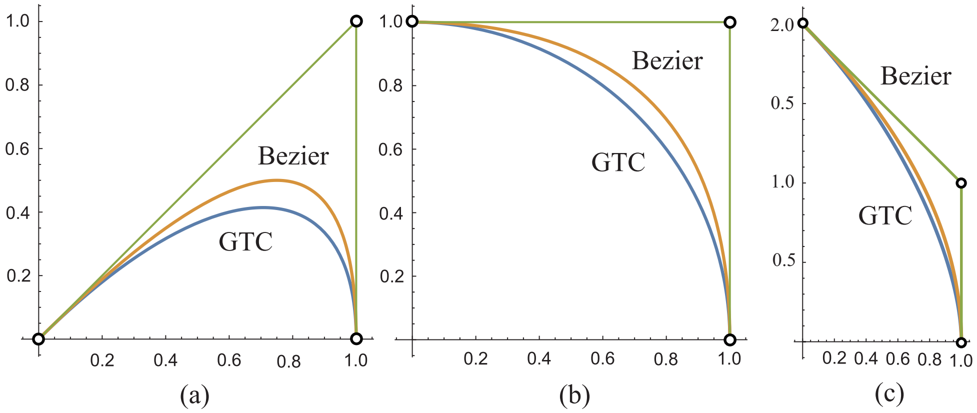


Figure 2: Examples of linear GT curves with quadratic Bézier curves.

4 RATIONAL QUADRATIC BÉZIER CURVE

It is very common to represent a circular arc by a quadratic rational Bézier curve as

$$\boldsymbol{C}(t) = \frac{(1-t)^2\boldsymbol{P}_0 + 2(1-t)t\sigma\boldsymbol{P}_1 + t^2\boldsymbol{P}_2}{(1-t)^2 + 2(1-t)t\sigma + t^2} \tag{15}$$

where σ is a weight of $\boldsymbol{P}_1$. For example when $\boldsymbol{P}_0 = (-1, a)$, $\boldsymbol{P}_1 = (0,0)$ and $\boldsymbol{P}_2 = (1, a)$ for a given a, if $\sigma = 1/\sqrt{a^2+1}$ the curve becomes a circular arc.

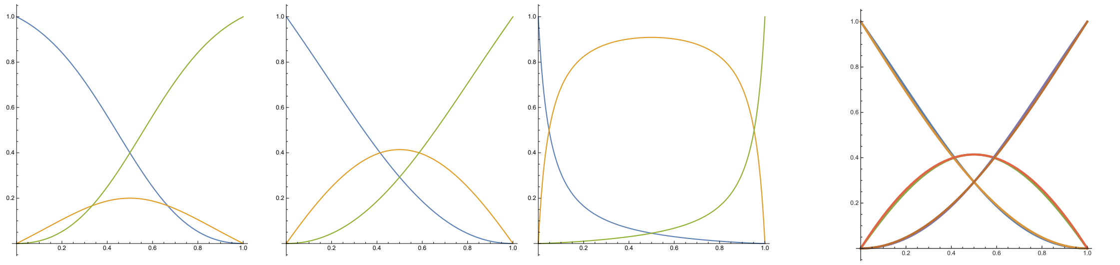
Hence we define a blending function $w(t)$ as follows:

$$w(t) = \frac{t^2}{(1-t)^2 + 2(1-t)t\sigma + t^2} \tag{16}$$

For this basis, the following equations is satisfied:

$$v(t)^2 = 4\sigma^2 u(t)w(t) \tag{17}$$

Figure 3(a) shows graphs of $\{u(t), v(t), w(t)\} = \{w(1-t), 1-w(1-t)-w(t), w(t)\}$ for $\sigma = 1/4$, $1/2$, $1/\sqrt{2}$, 2 and 10. By increasing σ, a curve defined by these basis functions approaches to a polyline connecting its control points.



σ=1/4 σ=1/sqrt(2) σ=10

(a) (b)

Figure 3: (a) Rational quadratic Bernstein basis functions, (b) Comparison between the rational quadratic Bernstein basis functions and $\{1-\sin(\pi t/2), \sin(\pi t/2)+\cos(\pi t/2)-1, 1-\cos(\pi t/2)\}$

Note that if $\sigma = 1$, since the basis becomes that of the non-rational quadratic Bernstein basis, $\alpha = 4$. If $\sigma = 1/\sqrt{2}$, $\alpha = 2$. However $w(t) \neq 1-\cos(\pi t/2)$. Figure 3(b) compares these two basis functions and they are very similar, but not indentical.

Since there are two types of the bases whose $\alpha = 2$, the conditions

$$\{1-w(t)-w(1-t)\}^2 = \alpha w(t)w(1-t) \tag{18}$$

for a given constant $\alpha > 0$ with $w(0) = 0$, $w(1) = 1$ and $dw(0)/dt = 0$ will not determine function $w(t)$ uniquely.

Notice that when $t = 1/2$, from the following equation:

$$\begin{aligned} (1-2w(\frac{1}{2}))^2 &= \alpha w(\frac{1}{2})^2 \\ (4-\alpha)w(\frac{1}{2})^2 - 4w(\frac{1}{2}) + 1 &= 0 \end{aligned} \tag{19}$$

When $\alpha = 4$, $w(1/2) = 1/4$. Since $0 < w(1/2) < 1$, when $\alpha < 4$, $w(1/2) = (2-\sqrt{\alpha})/(4-\alpha)$ and when $\alpha > 4$, $w(1/2) = (\sqrt{\alpha}-2)/(\alpha-4)$. Therefore although the basis functions are different, if they have the same α value, when $t = 1/2$, the values of these basis functions are exactly the same.

5 UNIQUENESS THEOREM OF THE SHAPE OF THE CURVE

We will prove a theorem called uniqueness theorem of the shape of the curve. We assume that for $0 \leq t \leq 1$ a curve $\boldsymbol{C}(t)$ is defined by three control points $\boldsymbol{P}_0$, $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ as

$$\boldsymbol{C}(t) = u(t)\boldsymbol{P}_0 + v(t)\boldsymbol{P}_1 + w(t)\boldsymbol{P}_2 \tag{20}$$

where $0 \leq w(t) \leq 1$, $0 \leq v(t) \leq 1$ and

$$\begin{aligned} u(t)+v(t)+w(t) &= 1 \\ u(t) &= w(1-t) \\ w(0) &= 0 \\ w(1) &= 1 \\ \frac{dw(t)}{dt} &> 0 \quad \text{for} \quad 0 < t < 1 \end{aligned} \tag{21}$$

If there is such a constant α that

$$v(t)^2 = \alpha u(t)w(t) \tag{22}$$

for $0 \leq t \leq 1$, then the following theorem is satisfied:

Theorem 1. *Uniqueness Theorem: The shape of the curve $\boldsymbol{C}(t)$ is determined by α exclusively and it does not depend on the basis functions $\{u(t), v(t), w(t)\}$ which are used to define the curve.*

Proof. For a given value $w_0 = w(t_0)$, $0 \leq w_0 \leq 1$, let $u_0 = u(t_0)$. Since $v(t) = 1 - u(t) - w(t)$,

$$(1 - u_0 - w_0)^2 = \alpha u_0 w_0 \tag{23}$$

Hence

$$u_0 = \frac{(\alpha - 2)w_0 + 2 - \sqrt{\alpha w_0((\alpha - 4)w_0 + 4)}}{2} \tag{24}$$

Since u_0 is uniquely determined by w_0, the location of the point $\boldsymbol{C}(t_0)$ is also uniquely determined because $\{u(t), v(t), w(t)\}$ are barycentric coordinates of triangle $\boldsymbol{P}_0\boldsymbol{P}_1\boldsymbol{P}_2$. By changing t from 0 to 1, $w(t)$ also increases from 0 to 1 and the shape of the curve $\boldsymbol{C}(t)$ is also completely determined. □

Figure 4 shows u_0 for $0 < w_0 < 1$ and $0 < \alpha < 10$

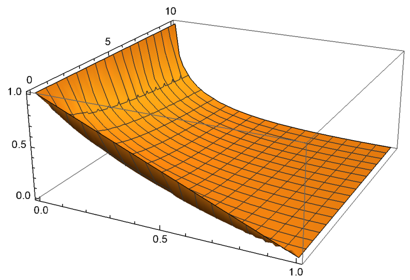


Figure 4: u_0 for $0 < w_0 < 1$ and $0 < \alpha < 10$

6 RATIONAL GENERALIZED TRIGONOMETRIC CURVE

Similar to the rational quadratic Bézier curve, with weight ω we define the rational linear generalized trigonometric curve as follows:

$$\boldsymbol{C}(t) = \frac{u(t)\boldsymbol{P}_0 + v(t)\omega\boldsymbol{P}_1 + w(t)\boldsymbol{P}_2}{u(t) + \omega v(t) + w(t)} \tag{25}$$

$$= u_r(t)\boldsymbol{P}_0 + v_r(t)\boldsymbol{P}_1 + w_r(t)\boldsymbol{P}_2 \tag{26}$$

where

$$u_r(t) = \frac{1 - S}{u(t) + v(t)\omega + w(t)}, \tag{27}$$

$$v_r(t) = \frac{S + C - 1}{u(t) + v(t)\omega + w(t)}, \tag{28}$$

$$w_r(t) = \frac{1 - C}{u(t) + v(t)\omega + w(t)}. \tag{29}$$

Then

$$v_r(t)^2 = 2\omega^2\, u_r(t)\, w_r(t) \tag{30}$$

Therefore by comparing equations (17) and (30), and applying Uniquness theorem, when $\omega = \sqrt{2}\sigma$, the shapes of the linear generalized trigonometric curve and the quadratic Bézier curve are identical although their parametrizations are different. Therefore the rational linear generalized trigonometric curve can represent an arbitrary elliptical arc as well as arbitrary arcs of parabola and hyperbola. Please refer to [4] about conics as rational quadratics. Furthermore by the same reason, if we rationalize generalized hyperbolic curve and splines in tension, they can represent an arbitrary elliptical arc as well as arbitrary arcs of parabola and hyperbola.

7 CONCLUSIONS

We has shown that the linear generalized trigonometric curve defined by three control points generates an elliptical arc, but cannot generate an arbitrary elliptic curve. Hence, we have rationalized it to express an arbitrary elliptical arc as well as arbitrary arcs of parabola and hyperbola. By the same reason, we have shown that the rational generalized hyperbolic curve and rational splines in tension can represent an arbitrary elliptical arc as well as arbitrary arcs of parabola and hyperbola. In the future research we will investigate other properties of these rationalized curves.

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Kenjiro T. Miura, http://orcid.org/0000-0001-9326-3130
R.U. Gobithaasan, http://orcid.org/0000-0003-3077-8772
Md Yushalify Misro, http://orcid.org/0000-0001-7869-0345
Tadatoshi Sekine, http://orcid.org/0000-0003-1813-822X
Shin Usuki, http://orcid.org/0000-0003-1165-5507

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