Curve Fitting Using Generalized Fractional Bézier Curve

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Abstract. Curve fitting is essential, especially when dealing with a function that can fit data with a high degree of smoothness. Bézier curve is one of the most widely utilized tools in the curve fitting process due to its beneficial properties. However, the shape of the curve cannot be changed without modifying its control points. Thus, the Bézier curve possesses limited flexibility and adjustability. Furthermore, shifting or altering control points to get the ideal shape is time-consuming. As a result, in this research, the generalized fractional Bézier curve will be used to overcome those limitations. The curve fitting procedure will become easier because the generalized fractional Bézier curve contains shape and fractional parameters, and the image outline can be built using the same degree of curve. Various levels of continuity, particularly fractional continuity, will be used. This fractional continuity allows the position of common points or connected joints along the first curve to be changed, offering additional flexibility in common point adjustment. As a result, the generalized fractional Bézier curve is a better choice for curve fitting.

Keywords: shape parameters, fractional parameter, fractional continuity, generalized fractional Bézier curve, curve fitting
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1 INTRODUCTION

The universe and its natural phenomena carries wondrous and exquisite shapes, which have long inspired scholars to study their mathematical representation and characteristics. These one-of-a-kind shapes can be created by combining several curves. Finding an exact depiction of a desired object, on the other hand, is challenging. Hence, an approximation curve or a free-form curve is utilized. The curve fitting method refers to the process of generating the forms of desired objects that have the best fit to a set of data points.

Curve fitting is a critical issue in a variety of areas, including computer-aided geometric design, manufacturing, medical imaging, and animation [10]. Bézier curve is one of the several free-form curves used in the curve fitting technique that are used to create shapes due to excellent geometric characteristics [4, 15]. [18] used piecewise G^1 continuity of cubic Bézier curves for curve fitting. [11] proposed a fitting procedure that is controlled by the displacement of a given curve point for data fitting of the Bézier curve. [14] applied the Differential Evolution algorithm to optimize the Bézier curve in curve fitting. Bézier curves are employed to provide data regarding the shapes of the leaves [19]. Bézier curve had also been applied in curve fittings of discrete data using the Clonal Selection Algorithm in the Artificial Immune Systems [9]. [5] discussed the potential of curve fitting as a tool in image extraction. The composite Bézier-like curves and blended cubic splines are also used in data fitting on manifolds [6].

The process of curve fittings can be made simpler by increasing the curve's flexibility and adaptability. However, the flexibility and adaptability of the classical Bézier curves are limited. To increase flexibility and adaptability, researchers created a variety of aesthetic Bézier curves. One of the upgraded versions of classical Bézier curves is the rational Bézier curves. The weight factors in rational Bézier curves allow for shape modification without modifying the control points, thus improving flexibility. Another kind of aesthetic Bézier curves such as the trigonometric Bézier curves with shape parameters is also developed. [2, 3, 7, 13] developed generalized trigonometric, general hybrid trigonometric, cubic trigonometric, and quintic trigonometric Bézier curves with shape parameters respectively. The implementation of shape parameters in the Bézier curve added flexibility to the curve, especially in the curve fitting process. [1] applied the quintic trigonometric Bézier curves with two shape parameters in the curve fitting of several images.

In this paper, the generalized fractional Bézier curve with shape parameters by [17], will be implemented in the curve fitting process. The generalized fractional Bézier curve has two types of parameters to improve curve flexibility and adjustability, known as shape and fractional parameters. The fractional parameter is the critical parameter that is responsible for controlling curve adjustability [16]. Due to the fractional parameter, an improved version of geometric and parametric continuity is developed. It is called as fractional continuity. This fractional continuity enables the joined point or common point of the connected curves to be moved along the first curve. Hence, in this paper, the utilization of shape and fractional parameters with fractional continuity will be applied in the curve fitting of an image.

The work is organized as follows. The generalized fractional Bézier basis curve will be introduced in Section 2. The geometric effect of shape parameters and fractional parameters on the curve will be also be briefed. Next, in Section 3, the definition and theorem regarding the fractional continuity for generalized fractional Bézier curve will be discussed. In Section 4, the generalized fractional Bézier curve will be applied in the curve fitting process. Several degrees of fractional continuity up to degree two will be used. The curvature comb will be added to see the smoothness of the connected curves. Last but not least, in Section 5, the conclusion and recommendation for future work will be addressed.

2 GENERALIZED FRACTIONAL BÉZIER CURVE

2.1 Generalized Fractional Bézier Basis Functions

Definition 2.1 (Generalized fractional Bézier basis functions). *For $t \in [0,1]$ and $v \geq 0$, the following function is defined as generalized fractional Bézier curve basis functions of degree n with n shape parameters:*

$$\bar{f}_{i,n}(t) = f_{i,n}(t)\Big(1 + \frac{\alpha_i}{n-i+1}(1 - D_t^{-v}(t)) + \big(-\frac{\alpha_{i+1}}{i+1}(D_t^{-v}(t))\Big), \quad -(n-i+1) < \alpha_i < i,\ \alpha_0 = \alpha_{n+1} = 0,\ i = 0, 1, ..., n, \tag{1}$$

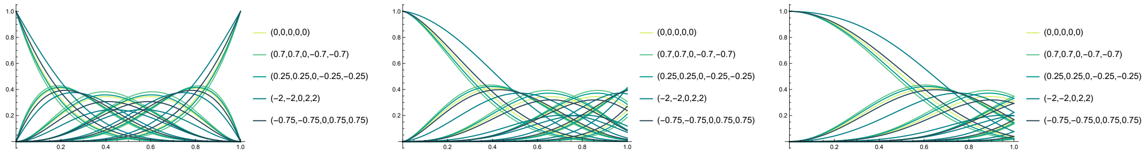
where $f_{i,n}(t) = \binom{n}{i}(1 - D_t^{-v}(t))^{n-i}(D_t^{-v}(t))^i$ and $D_t^{-v}(t) = \frac{1}{\Gamma(v+2)}t^{v+1}$.

Theorem 2.1 (Generalized fractional Bézier basis functions). *The generalized fractional basis functions with shape parameters have the following properties:*

1. *Degeneracy: When $v = 0$ and $\alpha_i = 0$ for $i = 1, 2, ..., n$, the basis functions become classical Bernstein basis functions.*
2. *Non-negativity: $\bar{f}_{i,n}(t) \geq 0, i = 0, 1, ...n$.*
3. *Partition of unity: $\sum_{i=0}^{n} \bar{f}_{i,n}(t) = 1$.*
4. *Symmetry: $\bar{f}_{i,n}(t) = \bar{f}_{n-i,n}(1-t), i = 0, 1, ..., n$ when $\alpha_i = -\alpha_{n-i+1}$ and $v = 0$.*

Proof. The proof can be seen in [17]. ■

Figure 1 shows the quintic fractional Bézier basis functions with multiple spectrum values of shape and fractional parameters, respectively. The figures illustrated the same as the classical Bernstein basis functions, especially when the shape and fractional parameters are set to zero.



(a) Quintic $v = 0$ (b) Quintic $v = 0.5$ (c) Quintic $v = 1$

Figure 1: Quintic fractional Bézier basis functions with multiple values of fractional and shape parameters.

2.2 Generalized Fractional Bézier Curve

Definition 2.2 (Generalized fractional Bézier curve). *The generalized fractional Bézier curve of n degree with n shape parameters is defined as follows:*

$$x(t; v, \alpha_1, \alpha_2, ..., \alpha_n) = \sum_{i=0}^{n} \bar{f}_{i,n}(t; v, \alpha_1, \alpha_2, ..., \alpha_n) P_i, \quad t \in [0, 1], \tag{2}$$

where $v \geq 0$, P_i for $i = 0, 1, ..., n$ is the set of control point in $\mathbb{R}^m$, $\alpha_0 = \alpha_{n+1} = 0$ and a few terms are defined below:

$$\bar{f}_{i,n}(t; v, \alpha_1, \alpha_2, ..., \alpha_n) = f_{i,n}(t)\Big(1 + \frac{\alpha_i}{n-i+1}(1 - D_u^{-v}(t)) - \frac{\alpha_{i+1}}{i+1}(D_u^{-v}(u))\Big),$$

$$f_{i,n}(t) = \binom{n}{i}(1 - D_t^{-v}(t))^{n-i}(D_t^{-v}(t))^i,$$

$$D_t^{-v}(t) = \frac{1}{\Gamma(v+2)} t^{v+1}.$$

Theorem 2.2 (Properties of generalized fractional Bézier curve). *The generalized fractional Bézier curve inherited all properties from classical Bézier curve with additional new properties such as the endpoint terminal, endpoint tangent, convex hull, geometric invariance, shape adjustable property, and fractional curve adjustable property.*

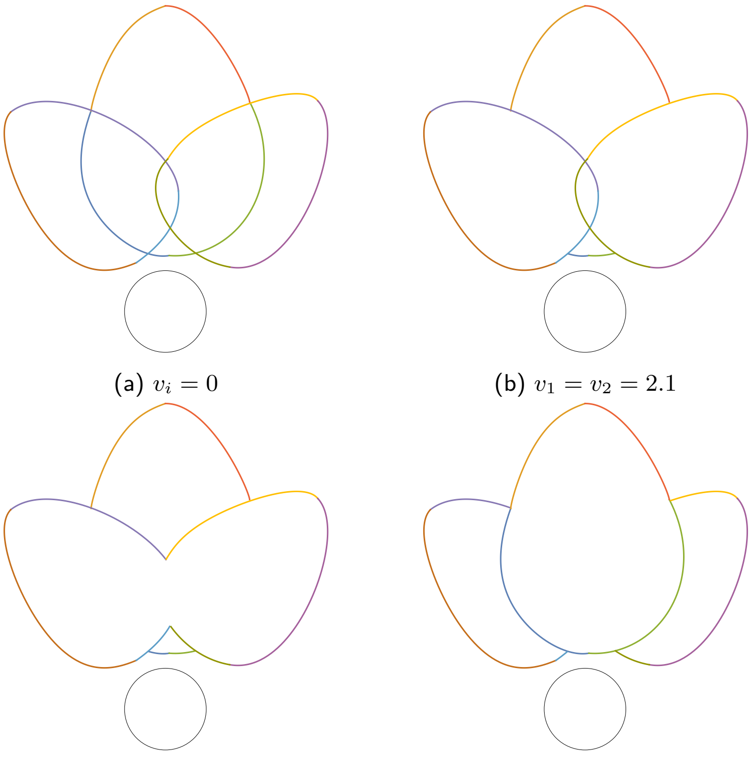
Proof. The proof for each properties can been seen in [17]. ■

2.3 Geometric Effect of Shape and Fractional Parameters

From Definition 2.2, the number of shape parameters depends on the degree of the curve. Each shape parameter controls the shape of the curve locally according to each control point. Generally, the shape parameter α_i will be associated with two control points; P_i and P_{i+1}. Increasing the value of α_i will cause the curve to move closer to control point P_{i+1}, while decreasing the value of α_i will cause the curve to move closer to control point P_i.

Every degree of curve will be embedded with one fractional parameter. This fractional parameter is used to control the curve adjustability. The higher the value of the fractional parameter, the shorter the curve. Example 2.1 shows the geometric effect of the fractional parameter to the modelling of a flower petal that contains overshooting and intersecting curves.

Example 2.1. *The modelling of a flower petal shape by multiple cubic fractional curves is shown in Figure 2. Ten cubic fractional Bézier curves are used. Each of the curves has a fractional parameter,* v_i $(i = 1, 2, \ldots, 10)$. v_1 *until* v_4 *are for the middle petal,* v_5 *until* v_7 *are for the left petal and the rest are for the right petal. Manipulating multiple values of fractional parameters will control the curve adjustability, thus produce different forms of petals.*



(a) $v_i = 0$

(b) $v_1 = v_2 = 2.1$

(c) $v_1 = v_2 = 2.1$, $v_5 = 0.31$, $v_7 = 1$, $v_{10} = 1.2$

(d) $v_5 = 1.04$, $v_7 = 2.18$, $v_8 = 1.25$ $v_{10} = 1.85$

Figure 2: Modelling of flower petal with variation of the fractional parameters.

3 CONTINUITY FOR GENERALIZED FRACTIONAL BÉZIER CURVE

In CAGD, constructing the complex curve can be done by using a higher degree of Bézier curves. However, this method is impractical and time-consuming. Hence, continuity is used to split the complex curve into numerous simpler low degrees of curves and connect them to form the complex curve. Therefore, the generalized fractional Bézier curve is described in the earlier section and the degree of the curve can be chosen freely. Consider two adjacent generalized fractional Bézier curves as follows:

$$x_1(t; v_1, \alpha_1, \alpha_2, ..., \alpha_n) = \sum_{i=0}^{n} \bar{f}_{i,n}(t; v_1, \alpha_1, \alpha_2, ..., \alpha_n) P_i, \ t \in [0,1], n \geq 3, \tag{3}$$

$$x_2(t; v_2, \beta_1, \beta_2, ..., \beta_n) = \sum_{j=0}^{m} \bar{f}_{j,m}(t; v_2, \beta_1, \beta_2, ..., \beta_n) Q_j, \ t \in [0,1], m \geq 3, \tag{4}$$

where $\bar{f}_{i,n}$ and $\bar{f}_{j,m}$ are generalized fractional Bézier basis functions of degree n and m. $\alpha_1, \alpha_2, ..., \alpha_n$ and $\beta_1, \beta_2, ..., \beta_n$ are the shape parameters, $P_i (i = 0, 1, ..., n)$ and $Q_j (j = 0, 1, ..., m)$ are the control points and v_1 and v_2 are fractional parameters for x_1 and x_2, respectively.

3.1 Fractional Continuity Constraints of Generalized Fractional Bézier Curve

The generalized fractional Bézier curve has a new kind of parameter called the fractional parameter. This fractional parameter has unlocked a new kind of continuity called fractional continuity. This fractional continuity overcomes the limitation of the parametric and geometric continuity which enable the curves to be connected at any arbitrary point along the first curve. The fractional continuity can be reduced to geometric and parametric continuity under special conditions.

Definition 3.1 (Fractional continuity for generalized fractional Bézier curve, F^r). *Consider two curves $x_1(t; v_1)$ on $t \in [a, b]$ and $x_2(t; v_2)$ on $t \in [a^*, b^*]$ with fractional parameters v_1 and v_2, respectively, where both curves have a degree of at least $r + 1$. The two curves are F^r continuous if the following condition is satisfied:*

$$\begin{cases} x_1(b; v_1) & = x_2(a^*; 0) \\ x_1'(b; v_1) & = \phi_1 x_2'(a^*; 0) \\ x_1''(b; v_1) & = \phi_1^2 x_2''(a^*; 0) + \phi_2 x_2'(a^*; 0) \\ & \vdots \\ x_1^{(r)}(b; v_1) & = \phi_1^r x_2^{(r)}(a^*; 0) + \phi_2^{r-1} x_2^{(r-1)}(a^*; 0) + ... + \phi_r x_2'(a^*; 0). \end{cases} \tag{5}$$

The range of the scalar factor, ϕ_i for $i = 0, 1, ..., r$ depends on the degree of continuity r. Generally, for F^r continuity, the range for the scale factors are $\phi_1 > 0$ and $\phi_i \in \mathbb{R}$ for $i = 2, 3, ..., r$.

Theorem 3.1 (Fractional continuity for generalized fractional Bézier curve, F^r). *Consider two same degrees of generalized fractional Bézier curves as in Equations (3) and (4), the necessary and sufficient conditions for fractional continuity at the joint points where $v_2 = 0$ are given by:*

1. *F^0 continuity:*

$$Q_0 = \sum_{i}^{n} \bar{f}_{i,n}(1; v_1, \alpha_1, \alpha_2, ..., \alpha_n) P_i. \tag{6}$$

2. F^1 *continuity:*

$$\begin{cases} Q_0 & = \sum_i^n \bar{f}_{i,n}(1; v_1, \alpha_1, \alpha_2, ..., \alpha_n) P_i, \\ Q_1 & = \frac{1}{\phi_1(n+\beta_1)} \Big(\frac{\mathrm{d}}{\mathrm{d}t} \big(\sum_i^n \bar{f}_{i,n}(t; v_1, \alpha_1, \alpha_2, ..., \alpha_n) P_i \big)_{t=1} \Big) + Q_0. \end{cases} \tag{7}$$

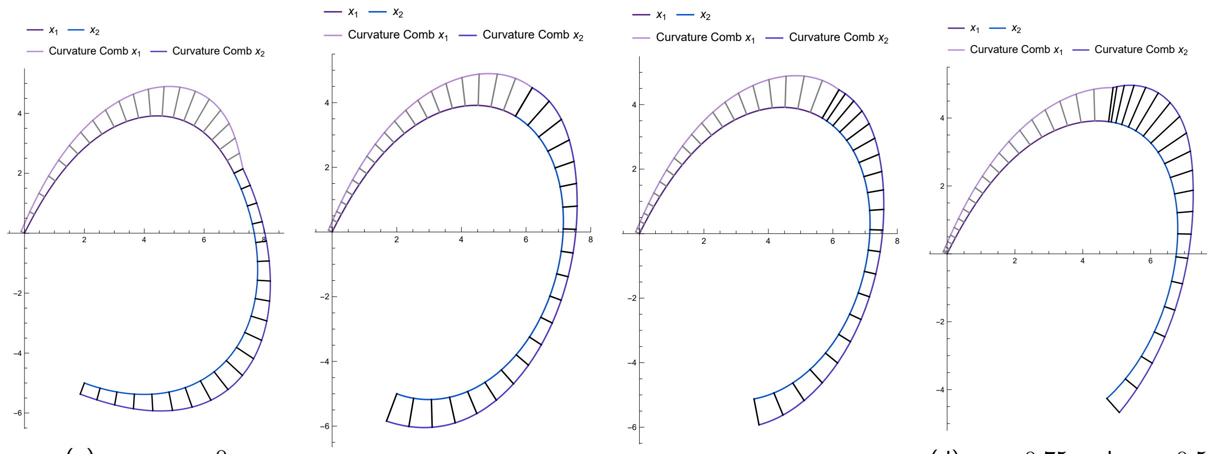
3. F^2 *continuity:*

$$\begin{cases} Q_0 & = \sum_i^n \bar{f}_{i,n}(1; v_1, \alpha_1, \alpha_2, ..., \alpha_n) P_i, \\ Q_1 & = \frac{1}{\phi_1(n+\beta_1)} \Big(\frac{\mathrm{d}}{\mathrm{d}t} \big(\sum_i^n \bar{f}_{i,n}(t; v_1, \alpha_1, \alpha_2, ..., \alpha_n) P_i \big)_{t=1} \Big) + Q_0, \\ Q_2 & = \frac{1}{n\phi_1^2(\beta_2+n-1)} \Big(\frac{\mathrm{d}^2}{\mathrm{d}t^2} \big(\sum_i^n \bar{B}_{i,n}(t; v_1, \alpha_1, \alpha_2, ..., \alpha_n) P_i \big)_{t=1}) \\ & \quad +(n\phi_1^2 - n^2\phi_1^2 + n\phi_2 - 2n\phi_1^2\beta_1 + \phi_2\beta_1) Q_0 \\ & \quad +(-2n\phi_1^2 + 2n^2\phi_1^2 - n\phi_2 + 2n\phi_1^2\beta_1 - \phi_2\beta_1 + n\phi_1^2\beta_2) Q_1 \Big). \end{cases} \tag{8}$$

If $v_1 = v_2 = 0$, then F^r continuity will revert back to geometric continuity, G^r. Thus, it can be reduced further to parametric continuity, C^r by setting the values of scalar factors, $\phi_1 = 1$ and $\phi_i = 0 (i = 2, 3, \ldots, r)$.

Proof. The proof for Theorem 3.1 can been seen in [17]. ■

Example 3.1. *Figure 3 depicts the F^2 continuity between two quartic fractional Bézier curves with variation values of fractional parameters. The control points for the first curve are $(0, 0)$, $(2, 4)$, $(4, 5)$, $(6, 4)$ and $(7, 2)$. The first, second, and third control points of the second curve can be obtained by using Theorem 3.1. The rest of the control points can be arbitrarily chosen by the user. In this example, the rest of the control points are $(6, -6.5)$ and $(2, -5)$. The shape parameters used are $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4) = (-0.75, -1, 0.5, -0.5, -0.5, -0.5, 0.9, 1.5)$. $\phi_1 = 0.9$ and $\phi_2 = 0.5$ are the scalar factors. The scale factor for the curvature comb, $d = 2.5$ is set.*



(a) $v_1 = v_2 = 0$ (b) $v_1 = 0.5$ and $v_2 = 0$ (c) $v_1 = 0.5$ and $v_2 = 0.3$ (d) $v_1 = 0.75$ and $v_2 = 0.5$

Figure 3: F^2 continuity between two quartic fractional Bézier curves with different values of fractional parameters.

From Figure 3, the fractional parameter can be used to control the position of the common point or joined point of the connected curve. This fractional continuity solves the restriction of the parametric and geometric continuity that only allows the curve to be joined at the endpoints. Hence, fractional continuity provides an extra advantage with the presence of an adjustability feature while maintaining the smoothness and continuity of the curve.

4 CURVE FITTING PROCESS OF IMAGE USING GENERALIZED FRACTIONAL BÉZIER CURVE

4.1 The Role of Fractional Parameter in the Curve Fitting Process

In modelling of curves, it is preferable to design a smooth curve. A curve is considered smooth if they have a minimum energy profile [8]. To achieve this minimum energy profile curve, a curve must have a low curvature profile. However, in practical application, the constructed curve may have a high curvature part and it is usually unavoidable. Having a high curvature profile is not preferable, especially in road design and curve fitting. In [12], a lower curvature profile road implies a higher maximum speed.

Hence, in this paper, a new technique called the "cut and combine" is proposed. This technique utilizes fractional parameters. The technique can minimize curvature values of the constructed curve by cutting unwanted parts that consist of high curvature values, and then combining it with the new low curvature profile curve. The process of this technique is shown in Figure 4. Based on Figure 4, the "cut and combine" technique can be used to construct a low curvature profile curve. Fractional continuity can be utilized to remove the high curvature part and be replaced it with new a new curve.

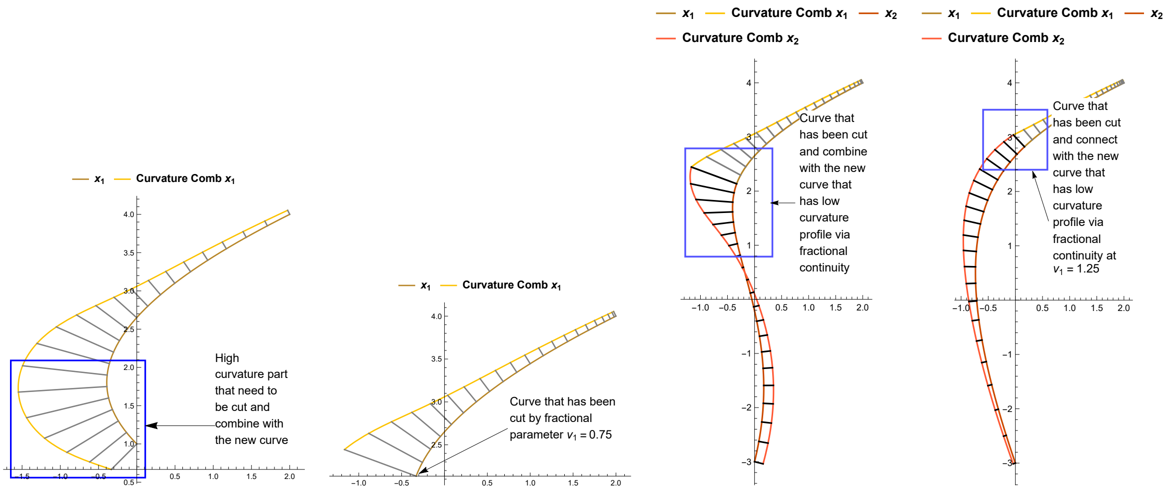


Figure 4: The process of "cut and combine" technique

4.2 Curve Fitting Process

Figure 5 shows an image of a leaf and the curve fitting process using several degrees of piecewise classical Bézier curves with their respective curvature comb. In this research, the curvature comb will be used to visualize the smoothness of the curve fitting process. The curvature comb demonstrates the changing state of the curvature from one point to another along the curve.

(a) Image of a leaf (b) Curve fitting process (c) Curvature comb of Figure 5b

Figure 5: The curve fitting process using several degrees of piecewise classical Bézier curves

Example 4.1. *Figure 6 illustrates the curve fitting process of Figure 5a using six piecewise cubic fractional Bézier curve. Note that only two joints are connected with F^0 continuity while the rest are C^0 continuity. The process of curve fitting is shown side by side with the respective curvature comb. The curvature comb is only shown for the curves that is connected with F^0 continuity. The double sided arrows indicate changes in the position of the curves' common point that have been connected using fractional continuity. Meanwhile, the dashed brown arrow represents the moving curve from the effect of shape parameters.*

Example 4.2. *In Figure 7, the process is the same as in Example 4.1 except two joints are connected with F^1 continuity.*

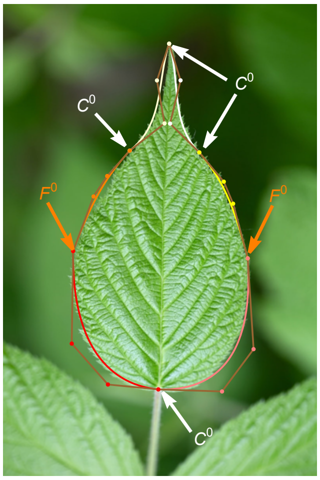
Example 4.3. *Figure 8 depicts the process of curve fitting of a leaf with F^2 continuity at two specific joints.*

In this paper, the application of curve fitting for animation will also be demonstrated. A game/anime character from Pokémon called Pikachu created by Game Freak and Creatures Inc. will be used to demonstrate the curve fitting. Figure 9 shows a picture of Pikachu's head.

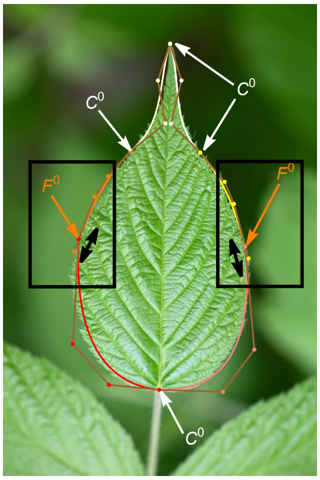
Example 4.4. *The method of Pikachu's curve fitting for F^0 continuity at six specified joints is depicted in Figure 10. Fifteen cubic fractional Bézier curves will be used in the curve fitting process. Figure 9 shows the image in decreased opacity for an easier curve fitting process.*

Example 4.5. *Figure 11 shows the method of curve fitting Pikachu for F^1 continuity at six specified joints. The process is the same as in Example 4.4.*

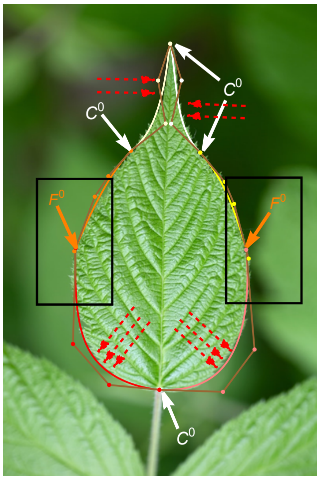
From Figure 6 - 11, a smoother shape can be constructed using the generalized fractional Bézier curve. The fractional and shape parameters can easily be adjusted and the parameter values can be chosen arbitrarily. Moreover, when the fractional continuity condition is satisfied, the fractional continuity will be formed automatically. Fractional continuity enables designers to shift the position of the common point of the connected curves along the first curve. Fractional continuity can be used as an alternative to the subdivision method, plus, it offers less computational time especially when involving higher degree curves.



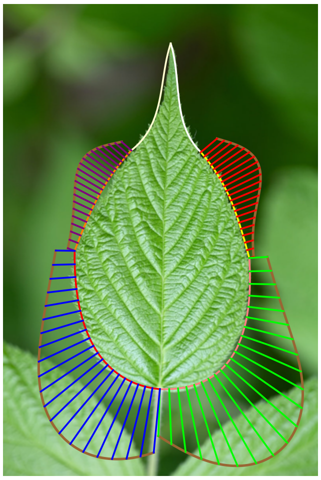
(a) Original curve without any adjustment.



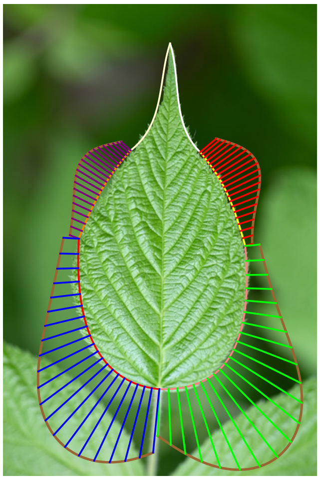
(b) Fractional parameters of F^0 joints are adjusted.



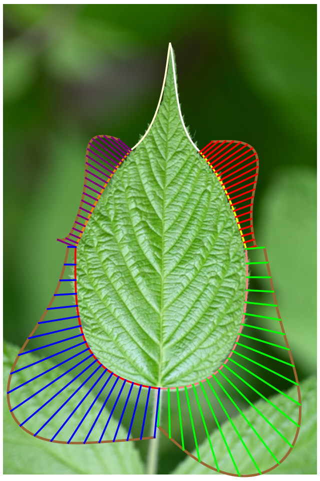
(c) Shape parameters are adjusted.



(d) Curvature comb of (a).



(e) Curvature comb of (b).

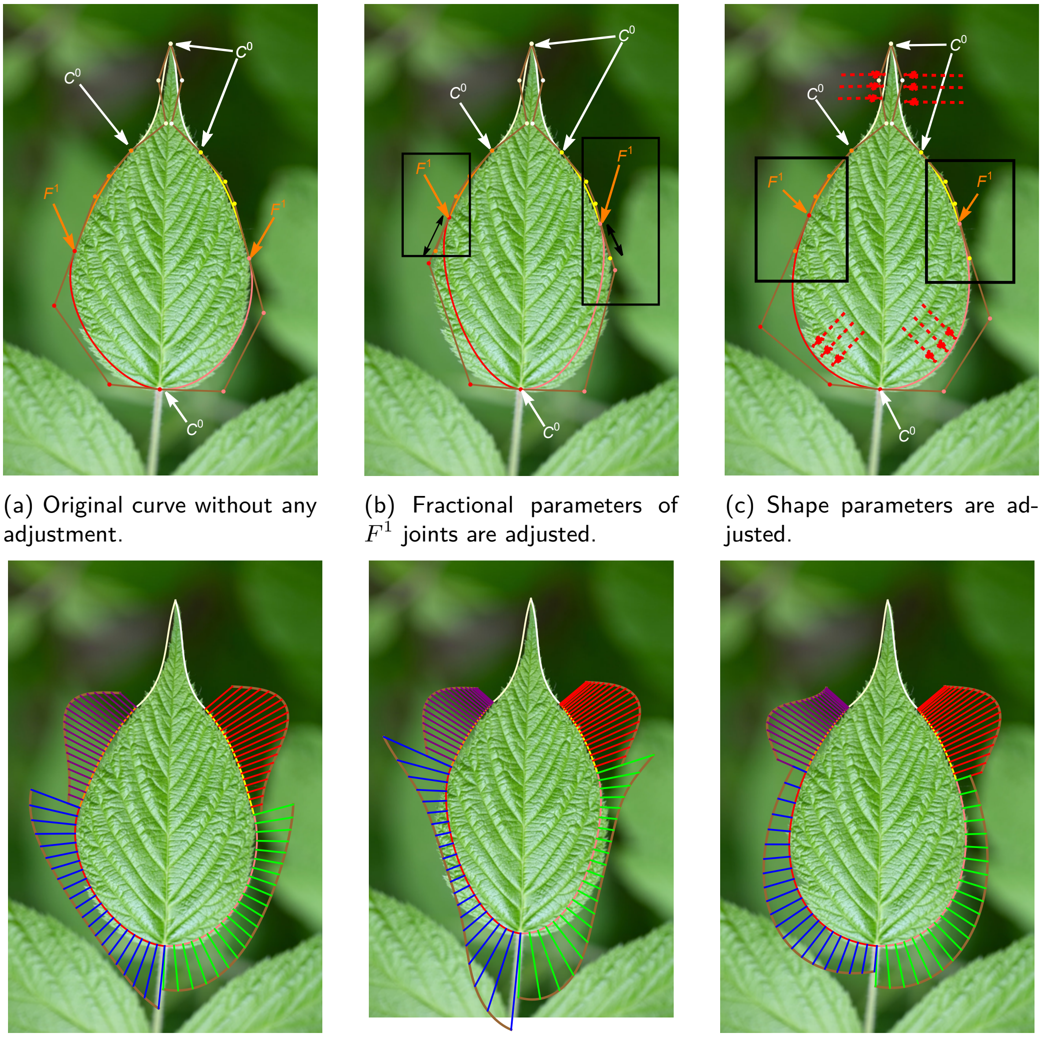


(f) Curvature comb of (c).

Figure 6: Curve fitting of a leaf with F^0 continuity.

Similar to other basis functions that consist of shape parameters, shape parameters will provide more control over the shape of the curve. Changing one control point of the Bézier curve can affect the whole shape of the curve. This will cause the curve fitting process using the classical Bézier curve to be inefficient compared to the generalized fractional Bézier curve.

However, there are some limitations of the generalized fractional Bézier curve. It has a complex computation compared to the classical Bézier curve. Fractional continuity also has a more complex equation compared to parametric/geometric continuity. Nevertheless, at the cost of complex computation, the generalized fractional Bézier curve provides easier manipulation of shape and adjustability of curve with the present of user interface in most updated software such as Mathematica. Therefore, it will be suitable to be used and implemented in other CAD/CAM softwares.



(a) Original curve without any adjustment.

(b) Fractional parameters of F^1 joints are adjusted.

(c) Shape parameters are adjusted.

(d) Curvature comb of (a).

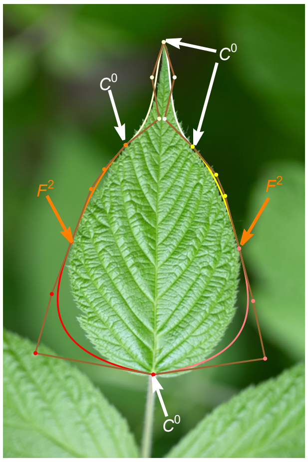
(e) Curvature comb of (b).

(f) Curvature comb of (c).

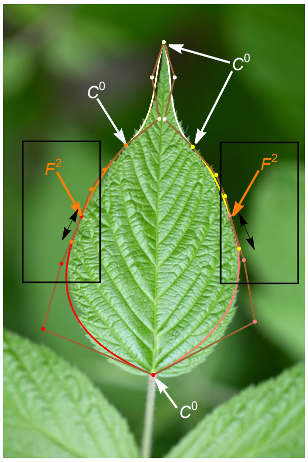
Figure 7: Curve fitting of a leaf with F^1 continuity.

5 CONCLUSIONS

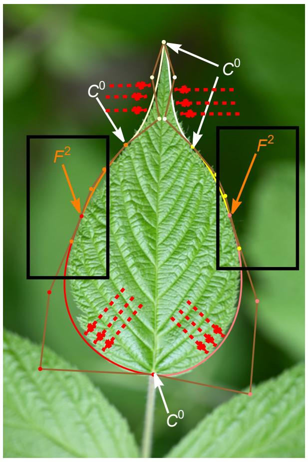
In this paper, the generalized fractional Bézier curve is used in the curve fitting process. Shape and fractional parameters help to make the curve fitting process faster and easier. The fractional continuity is a useful tool to replace the higher curvature part with the minimum value of curvature while maintaining the smoothness of the curve. In terms of computational time, fractional continuity and fractional parameter helped to make the generalized fractional Bézier curve a superior technique over the subdivision method. Therefore, the generalized fractional Bézier curve is an excellent tool to apply in curve fittings. It is suggested for future works to apply the generalized fractional Bézier curve in optimization methods, such as genetic algorithm and particle swarm optimization in data fitting.



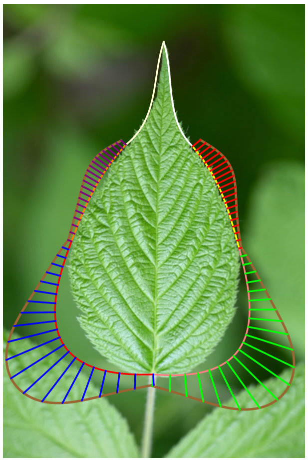
(a) Original curve without any adjustment.



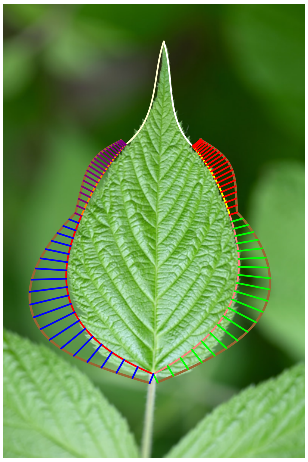
(b) Fractional parameters of F^2 joints are adjusted.



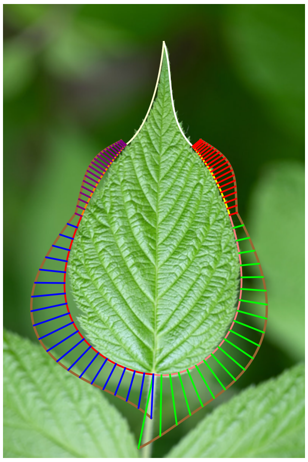
(c) Shape parameters are adjusted.



(d) Curvature comb of (a).



(e) Curvature comb of (b).



(f) Curvature comb of (c).

Figure 8: Curve fitting of a leaf with F^2 continuity.

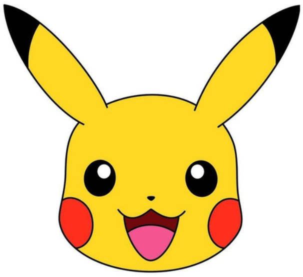
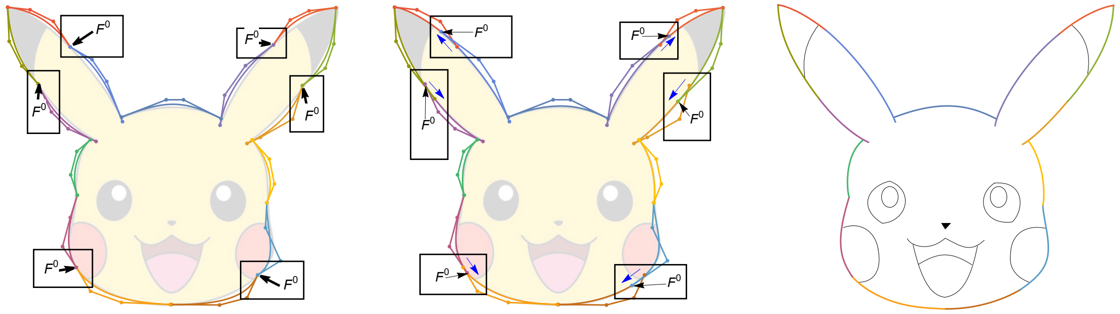
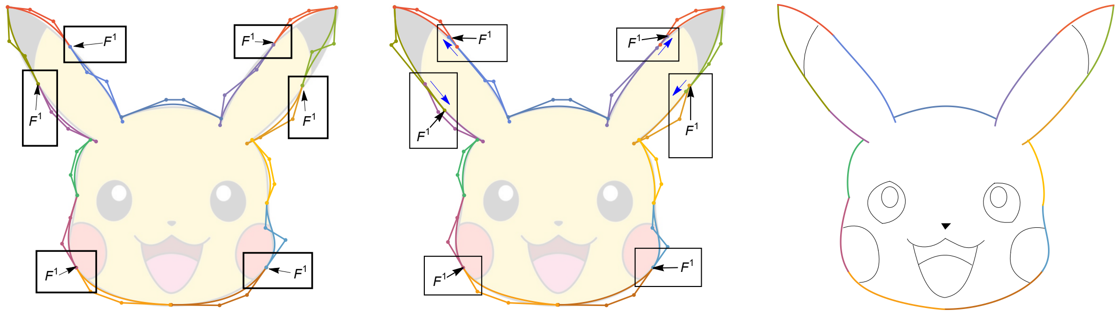


Figure 9: Image of Pikachu. (Source: https://www.pinterest.pt/pin/318207529927050146/)



(a) Original curve without any adjustment. (b) Fractional and shape parameters are adjusted. (c) Curve fitting process without original image.

Figure 10: Curve fitting process of Pikachu using F^0 continuity.



(a) Original curve without any adjustment. (b) Fractional and shape parameters are adjusted. (c) Curve fitting process without the original image.

Figure 11: Curve fitting process of Pikachu using F^1 continuity.

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