Generalization of the Shape Uniqueness Theorem for Free-form Curves

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Abstract. The shape uniqueness theorem for free-form curves shows the conditions on which the shapes of two parametric curves defined by three control points are identical although their parametrization may be different. According to this theorem, even though their blending functions look different, the curves become identical by reparametrization under some conditions on their blending functions. In this paper, we will extend this theorem for curves defined by four or more control points and show several examples of applications of the theorem.

Keywords: uniqueness theorem, reparametrization, cubic curve, polynomial curve, trigonometric curve, Gobithaasan-Miura's recursive algorithm
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1 INTRODUCTION

The shape uniqueness theorem for free-form curves shows the conditions on which the shapes of two parametric curves defined by three control points are identical although their parametrization may be different [6]. According to this theorem, even though their blending functions look different, the curves become identical by reparametrization under some conditions on their blending functions.

A lot of researches have been done on the blending functions of free-form curves so far and many types of free-from curves are available for curve designers. These designers must be confused on which type of curve should be used for their design. We hope that the shape uniqueness theorem for free-form curves will help the designers classify and categorize types of curves and select the most suitable one for their design purposes because it identifies the curves which superficially look different but represent the same shape.

In this paper, we will extend the shape uniqueness theorem for curves that are defined by four or more control points and shows several examples of applications of the theorem.

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2 IDENTICAL SHAPE OF FREE-FORM CURVES

Identical shape of two parametric curves is defined as follows [2]:

Definition 1. *For two parametric curves $\boldsymbol{r}: I \to R^3$ and $\tilde{\boldsymbol{r}}: \tilde{I} \to R^3$, there exists a C^∞ function $\phi: I \to \tilde{I}$, 1) ϕ is a one to one and onto mapping from I to $\tilde{I}$. 2) ϕ is strictly increasing. 3) For all $t \in I$, $\tilde{\boldsymbol{r}}(\phi(t)) = \boldsymbol{r}(t)$. We say that $\boldsymbol{r}$ and $\tilde{\boldsymbol{r}}$ define the same curve or their shapes are identical.*

Then $\tilde{\boldsymbol{r}}((\phi(t))$ is called reparametrization of $\boldsymbol{r}(t)$. We explain the meaning of the above definition using a rather trivial example. Figure 1 shows two types of the parametrization of straight line segment $\boldsymbol{P}_0\boldsymbol{P}_1$. The line segment is given by $\boldsymbol{C}(t) = (1-t)\boldsymbol{P}_0 + t\boldsymbol{P}_1$. The other parametrization is $\boldsymbol{C}(t) = \cos\frac{\pi}{2}t\boldsymbol{P}_0 + (1-\cos\frac{\pi}{2}t)\boldsymbol{P}_1$. Geometrically they represent the same curve: line segment $\boldsymbol{P}_0\boldsymbol{P}_1$ although they look different. We write the following theorem for completeness. It is trivial, but it makes clear the role of the uniqueness theorem of the shape of the curve defined by control points.

Theorem 1. *Uniqueness Theorem of the Shape of the Curve Defined by Two Control Points: The shape of the curve $\boldsymbol{C}(t)$ defined by two control points does not depend on its blending functions and the start and end points determine its shape uniquely.*

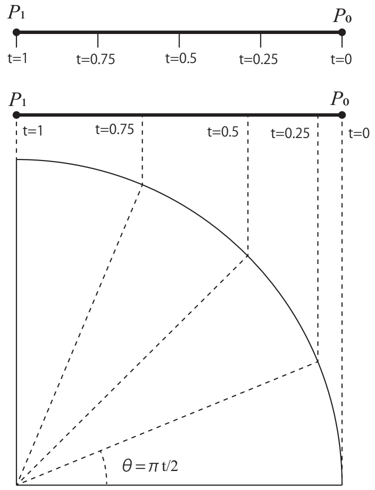

Figure 1: Two types of parametrization of the straight line segment

3 Uniqueness Theorem of the Shape of the Curve Defined by Three Control Points: [6]

In this paper, we assume that for $0 \le t \le 1$, a curve $\boldsymbol{C}(t)$ is defined by three control points $\boldsymbol{P}_0$, $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ as

$$\boldsymbol{C}(t) = u(t)\boldsymbol{P}_0 + v(t)\boldsymbol{P}_1 + w(t)\boldsymbol{P}_2 \tag{1}$$

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where $w(0) = 0$, $w(1) = 1$, $0 \leq u(t), w(t), v(t) \leq 1$ and

$$\begin{aligned} u(t) + v(t) + w(t) &= 1 \\ w(0) &= 0 \\ w(1) &= 1 \\ \frac{dw(t)}{dt} &> 0 \quad \text{for} \quad 0 < t < 1 \end{aligned} \tag{2}$$

We have removed the condition that $u(t) = w(1-t)$ from the original definition [6] since the theorem is still satisfied. If there is such a constant α that

$$v(t)^2 = \alpha u(t) w(t) \tag{3}$$

for $0 \leq t \leq 1$, then the following theorem is satisfied:

Theorem 2. *Uniqueness Theorem: The shape of the curve $\boldsymbol{C}(t)$ is determined by α exclusively and it does not depend on the basis functions $\{u(t), v(t), w(t)\}$ which are used to define the curve.*

Proof. For a given value $w_0 = w(t_0)$, $0 \leq w_0 \leq 1$, let $u_0 = u(t_0)$. Since $v(t) = 1 - u(t) - w(t)$,

$$(1 - u_0 - w_0)^2 = \alpha u_0 w_0 \tag{4}$$

Hence

$$u_0 = \frac{(\alpha - 2)w_0 + 2 \pm \sqrt{\alpha w_0((\alpha - 4)w_0 + 4)}}{2} \tag{5}$$

u_0 should satisfy $0 \leq u_0 \leq 1 - w_0$. From $u_0 \leq 1 - w_0$, by simple calculation the solution with $+$ sign before the square root is found not to be adequate. The solution with $-$ sign satisfies both of the conditions of $0 \leq u_0$ and $u_0 \leq 1 - w_0$. Therefore u_0 is uniquely determined by w_0 and the location of the point $\boldsymbol{C}(t_0)$ is also uniquely determined because $\{u(t), v(t), w(t)\}$ are barycentric coordinates of triangle $\boldsymbol{P}_0\boldsymbol{P}_1\boldsymbol{P}_2$. By changing t from 0 to 1, $w(t)$ also increases from 0 to 1 and the shape of the curve $\boldsymbol{C}(t)$ is also completely determined. Q.E.D. □

Then $u(t) = u(w(t))$, $v(t) = v(w(t))$, and $w = w(t)$ are reparametrized blending functions. For example, the blending functions of quadratic Bézier curve $u(t) = (1-t)^2$, $v(t) = 2(1-t)t$, and $w(t) = t^2$ give $\alpha = 4$ and $u(w(t)) = (1 - \sqrt{w(t)})^2$, $v(w(t)) = 2(1 - \sqrt{w(t)})\sqrt{w(t)}$.

Figure 2 shows u_0 for $0 < w_0 < 1$ and $0 < \alpha < 10$.

4 GENERALIZATION - THE CASE WHERE GOBITHAASAN-MIURA'S RECURSIVE ALGORITHM IS SATISFIED

In this section, we assume that the blending functions satisfy the recursive relationship of generalized trigonometric basis [5] , which yields Gobithaasan-Miura's recursive algorithm [3, 4]. The original algorithm is for the cubic, but we extend it for any degrees. For example, for the quartic case,

$$\begin{aligned} \boldsymbol{C}(t) &= u(u\boldsymbol{P}_0 + v\boldsymbol{P}_1 + w\boldsymbol{P}_2 + x\boldsymbol{P}_3) \\ &\quad + v(u\boldsymbol{P}_1 + v\boldsymbol{P}_2 + w\boldsymbol{P}_3 + x\boldsymbol{P}_4) \\ &\quad + w(u\boldsymbol{P}_2 + v\boldsymbol{P}_3 + w\boldsymbol{P}_4 + x\boldsymbol{P}_5) \\ &\quad + x(u\boldsymbol{P}_3 + v\boldsymbol{P}_4 + w\boldsymbol{P}_5 + x\boldsymbol{P}_6) \\ &= u^2\boldsymbol{P}_0 + 2uv\boldsymbol{P}_1 + (2uw + v^2)\boldsymbol{P}_2 + 2(ux + vw)\boldsymbol{P}_3 + (2vx + w^2)\boldsymbol{P}_4 + 2wx\boldsymbol{P}_5 + x^2\boldsymbol{P}_6 \end{aligned}$$

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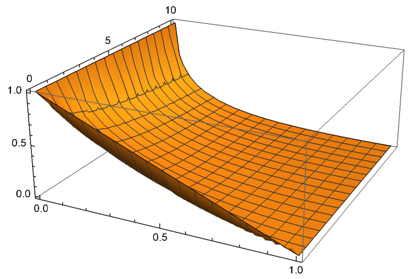

Figure 2: u_0 for $0 < w_0 < 1$ and $0 < \alpha < 10$

where the blending functions u, v, w, and x of parameter t are assumed to satisfy partition of unity. Hence for an arbitrary $t \in [0, 1]$,

$$u + v + w + x = 1 \tag{6}$$

is satisfied.

For the curve to be represented by seven control points with seven blending functions, the following equations must be satisfied:

$$\begin{aligned} v^2 &= \alpha uw && (7) \\ w^2 &= \beta vx && (8) \\ vw &= \gamma ux && (9) \end{aligned}$$

where $\alpha > 0$, $\beta > 0$, and $\gamma > 0$ are constants that are independent from parameter t. However, the product of both sides of Eqs.(7) and (8) yields

$$\begin{aligned} v^2w^2 &= \alpha\beta uvwx \\ vw &= \alpha\beta ux && (10) \end{aligned}$$

Therefore

$$\gamma = \alpha\beta \tag{11}$$

When α and β satisfy Eqs.(7) and (8), respectively, Eq.(9) is automatically satisfied.

Therefore, if the blending functions u, v, w and x satisfy the following conditions, for a given function x the other functions u, v, and w are uniquely determined. Thus we can elevate the degree and increase the number of control points of the shape uniqueness theorem.

$$u + v + w + x = 1,\ v^2 = \alpha uw,\ w^2 = \beta vx. \tag{12}$$

The function $x(t)$ satisfies the followings:

$$\begin{aligned} x(0) &= 0, \\ x(1) &= 1, \\ \frac{dx(t)}{dt} &> 0. && (13) \end{aligned}$$

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Theorem 3. *Shape Uniqueness Theorem of Higher Degree (#control points= 4): The shape of the curve $\boldsymbol{C}(t)$ is determined by α and β and it does not depend on the blending functions of use $\{u(t), v(t), w(t), x(t)\}$.*

Proof. For $x_0 = x(t_0)\,(0 \leq x_0 \leq 1)$, we assume that $u_0 = u(t_0)$, $v_0 = v(t_0)$, and $w_0 = w(t_0)$. From Eqs. (7) and (8),

$$v_0 = \alpha^{\frac{2}{3}}\beta^{\frac{1}{3}}u_0^{\frac{2}{3}}x_0^{\frac{1}{3}} \tag{14}$$

$$w_0 = \alpha^{\frac{1}{3}}\beta^{\frac{2}{3}}u_0^{\frac{1}{3}}x_0^{\frac{2}{3}} \tag{15}$$

Since $u_0 + v_0 + w_0 + x_0 - 1 = 0$,

$$u_0 + \alpha^{\frac{2}{3}}\beta^{\frac{1}{3}}u_0^{\frac{2}{3}}x_0^{\frac{1}{3}} + \alpha^{\frac{1}{3}}\beta^{\frac{2}{3}}u_0^{\frac{1}{3}}x_0^{\frac{2}{3}} + x_0 - 1 = 0 \tag{16}$$

Let the left side of the above equation be $f(u_0; x_0)$. When $x_0 = 0$,

$$f(u_0; 0) = u_0 - 1 \tag{17}$$

Hence $u_0 = 1$. When $x_0 = 1$,

$$f(u_0; 1) = u_0^{\frac{1}{3}}(u_0^{\frac{2}{3}} + \alpha^{\frac{2}{3}}\beta^{\frac{1}{3}}u_0^{\frac{1}{3}} + \alpha^{\frac{1}{3}}\beta^{\frac{2}{3}}) \tag{18}$$

Then $u_0 = 0$.

If we assume that $0 < x_0 < 1$,

$$f(0; x_0) = x_0 - 1 < 0$$
$$f(1; x_0) = \alpha^{\frac{2}{3}}\beta^{\frac{1}{3}}x_0^{\frac{1}{3}} + \alpha^{\frac{1}{3}}\beta^{\frac{2}{3}}x_0^{\frac{2}{3}} + x_0 > 0$$

Furthermore

$$\frac{\partial f(u_0; x_0)}{\partial u_0} = 1 + \frac{2}{3}\alpha^{\frac{2}{3}}\beta^{\frac{1}{3}}x_0^{\frac{1}{3}}u_0^{-\frac{1}{3}} + \frac{1}{3}\alpha^{\frac{1}{3}}\beta^{\frac{2}{3}}x_0^{\frac{2}{3}}u_0^{-\frac{2}{3}} > 0 \tag{19}$$

Hence for x_0, $f(u_0; x_0)$ is a continuous function of u_0 and strictly increasing. Since $f(0; x_0) < 0$ and $f(1; x_0) > 0$, For x_0, u_0 is determined such that $0 \leq u_0 \leq 1$. Similarly, v_0 and w_0 are determined uniquely from $v_0^2 = \alpha u_0 w_0$ and $w_0^2 = \beta v_0 x_0$ from Eqs.(14) and (15). $\{u(t), v(t), w(t), x(t)\}$ are barycentric coordinates of tetrahedron $\boldsymbol{P}_0\boldsymbol{P}_1\boldsymbol{P}_2\boldsymbol{P}_3$ and $\boldsymbol{C}(t_0)$ is uniquely determined. When t changes from 0 to 1, $x(t)$ changes 0 to 1 and the whole shape of the curve $\boldsymbol{C}(t)$ is determined completely. Q.E.D. □

Note that even when tetrahedron $\boldsymbol{P}_0\boldsymbol{P}_1\boldsymbol{P}_2\boldsymbol{P}_3$ is degenerated into a 2D plane, the shape of the curve is uniquely determined by barycentric coordinates.

4.1 An Application to the Rational Cubic Bézier Curve

It is well known that as a reparametrization of a rational Bézier curve of degree n, its weights w_i can be changed without changing the curve shape as follows [1]:

$$\hat{w}_i = c^i w_i; \quad i = 0, \ldots, n. \tag{20}$$

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where $c \neq 0$ is a constant. For example, when $c = \sqrt[n]{w_0/w_n}$, then if we subdivide all weights by w_0, we obtain $w_0 = w_n = 1$. When $n = 3$,

$$\begin{aligned} u(t) &= \frac{(1-t)^3 w_0}{f(t)}, \\ v(t) &= \frac{3(1-t)^2 t w_1}{f(t)}, \\ w(t) &= \frac{3(1-t)t^2 w_2}{f(t)}, \\ x(t) &= \frac{t^3 w_3}{f(t)}. \end{aligned} \tag{21}$$

where $f(t) = (1-t)^3 w_0 + 3(1-t)^2 t w_1 + 3(1-t)t^2 w_2 + t^3 w_3$. On these blending functions,

$$\begin{aligned} \alpha &= \frac{v(t)^2}{u(t)v(t)} = \frac{3w_1^2}{w_0 w_2} \\ \beta &= \frac{w(t)^2}{v(t)x(t)} = \frac{3w_2^2}{w_1 w_3} \end{aligned}$$

When $c = \sqrt[3]{w_0/w_3}$, $\hat{w}_0 = w_0$, $\hat{w}_1 = cw_1$, $\hat{w}_2 = c^2 w_2$, and $\hat{w}_3 = c^3 w_3$. Then

$$\begin{aligned} \frac{3\hat{w}_1^2}{\hat{w}_0 \hat{w}_2} &= \frac{3w_1^2}{w_0 w_2} \\ \frac{3\hat{w}_2^2}{\hat{w}_1 \hat{w}_3} &= \frac{3w_2^2}{w_1 w_3} \end{aligned}$$

are satisfied. Therefore, from the shape uniqueness theorem of higher degree (the number of control points $= 4$), we know the shape is unchanged. Note that when the number of control points $= 3$, a similar argument is satisfied. When $w_0 = w_3 = 1$ as "normalized", we obtain

$$\begin{aligned} \alpha &= \frac{3w_1^2}{w_2}, \\ \beta &= \frac{3w_2^2}{w_1} \end{aligned} \tag{22}$$

4.2 Shape Uniqueness Theorem for General Degrees

It is straightforward to extend the shape uniqueness theorem for the general degree n as follows: We assume that the curve is defined by $n+1$ control points using $n+1$ blending functions $b_i(t)$. These functions satisfy

$$\begin{aligned} \sum_{i=0}^{n} b_i(t) &= 1 \\ b_n(0) &= 0 \\ b_n(1) &= 1 \\ \frac{db_i(t)}{dt} &> 0 \quad \text{for} \quad 0 < t < 1 \end{aligned} \tag{23}$$

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Under these condtions, we add the following $n-1$ conditions:

$$\begin{aligned} b_1(t)^2 &= \alpha_1 b_0(t) b_2(t) \\ b_2(t)^2 &= \alpha_2 b_1(t) b_3(t) \\ &\cdots \\ b_{n-1}(t)^2 &= \alpha_{n-1} b_{n-2}(t) b_n(t) \end{aligned} \tag{24}$$

where α_i, $i = 1, 2, \ldots, n$ are constants which do not depend on parameter t. Note that we can apply this theorem of degree n to the rational Bézier curve of degree n as shown in the previous subsection.

5 GENERALIZATION - THE CASE WHERE GOBITHAASAN-MIURA'S RECURSIVE ALGORITHM IS NOT SATISFIED

In this section, we will deal with the more general case where Gobithaasan-Miura's recursive algorithm is NOT satisfied. At first, we will generalize the shape uniqueness theorem for the curves with 4 control points.

We define a curve $\boldsymbol{C}(t)$ $(0 \le t \le 1)$ with four control points $\boldsymbol{P}_i$, $i = 0, \cdots, 3$ as follows:

$$\boldsymbol{C}(t) = u(t)\boldsymbol{P}_0 + v(t)\boldsymbol{P}_1 + w(t)\boldsymbol{P}_2 + x(t)\boldsymbol{P}_3 \tag{25}$$

where $0 \le u(t),\ v(t),\ w(t),\ x(t) \le 1$. We assume that

$$\begin{aligned} & u(t) + v(t) + w(t) + x(t) = 1,\ x(0) = 0, x(1) = 1, \\ & \frac{dx(t)}{dt} > 0 \\ & f(u(t), v(t), w(t), x(t)) = 0,\ g(u(t), v(t), w(t), x(t)) = 0 \end{aligned} \tag{26}$$

Hence, functions $f(u, v, w, x)$ and $g(u, v, w, x)$ do not depend on t. The proof for this type of the shape uniqueness theorem depends on the actual $f(u, v, w, x)$ and $g(u, v, w, x)$ and we show some examples of the C^2 interpolating spline [7] in such a case.

5.1 A Class of C^2 Interpolating Splines

C^2 interpolating splines [7] was proposed as a class of spline curves in SIGGRAPH 2020 and interpolate given point sequence. Depending on the curve segment types, they are classified as 1) Quadratic Bézier curve, 2) Circular arc, 3) Elliptic arc, 4) Hybrid of circular and elliptic arcs.

They are defined as follows: $\boldsymbol{F}_i$ are interpolating functions passing through $\boldsymbol{p}_{i-1}$, $\boldsymbol{p}_i$, and $\boldsymbol{p}_{i+1}$. Interpolating functions $\boldsymbol{F}_i$ are

$$\boldsymbol{F}_i(0) = \boldsymbol{p}_{i-1} \tag{27}$$

$$\boldsymbol{F}_i(\frac{\pi}{2}) = \boldsymbol{p}_i \tag{28}$$

$$\boldsymbol{F}_i(\pi) = \boldsymbol{p}_{i+1} \tag{29}$$

Curve segments $\boldsymbol{C}_i$ consist of the blending of interpolating functions $\boldsymbol{F}_i$ and $\boldsymbol{F}_{i+1}$ passing through $\boldsymbol{p}_i$ and $\boldsymbol{p}_{i+1}$. Trigonometric functions are used as blending functions such that

$$\boldsymbol{C}_i(\theta) = \cos^2\theta \boldsymbol{F}_i(\theta + \frac{\pi}{2}) + \sin^2\theta \boldsymbol{F}_{i+1}(\theta) \tag{30}$$

where $\theta \in [0, \frac{\pi}{2}]$ is a normalized parameter value. For arbitrary blending functions, C^2-continuity of the curve is guaranteed.

Figure 3 shows open and closed curve examples of C^2 interpolating splines of quadratic Bézier type.

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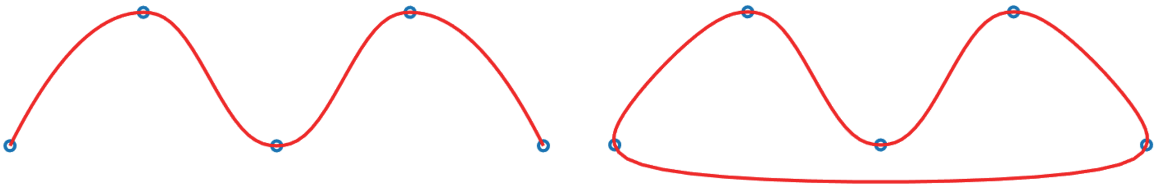

Figure 3: C^2 interpolating splines.

5.2 Bézier Type

When the C^2 interpolating spline adopts the quadratic Bézier curve segment, the blending functions, for example, from $\boldsymbol{p}_i$ to $\boldsymbol{p}_{i+1}$ are given by

$$u(t) = (1-t)^2 \tag{31}$$
$$v(t) = 2(1-t)t\cos^2(\frac{\pi}{2}t) \tag{32}$$
$$w(t) = 2(1-t)t\sin^2(\frac{\pi}{2}t) \tag{33}$$
$$x(t) = t^2 \tag{34}$$

because the control points of the curve segment passing $\boldsymbol{p}_{i-1}$, $\boldsymbol{p}_i$ and $\boldsymbol{p}_{i+1}$ are given by $\{\boldsymbol{P}_0, \boldsymbol{P}_1, \boldsymbol{P}_2\}$ and the one passing $\boldsymbol{p}_i$, $\boldsymbol{p}_{i+1}$ and $\boldsymbol{p}_{i+2}$ by $\{\boldsymbol{P}_1, \boldsymbol{P}_2, \boldsymbol{P}_3\}$. Note that $\boldsymbol{P}_0 = \boldsymbol{p}_i$ and $\boldsymbol{P}_3 = \boldsymbol{p}_{i+1}$. Then

$$u(t) + v(t) + w(t) + x(t) = 1 \tag{35}$$
$$(v(t) + w(t))^2 = 4u(t)x(t) \tag{36}$$
$$v(t)\sin^2(\frac{\pi}{2}t) = w(t)\cos^2(\frac{\pi}{2}t) \tag{37}$$

Note that in this case $f(u(t), v(t), w(t), x(t))$ and $g(u(t), v(t), w(t), x(t))$ in Eq.(26) are defined as

$$\begin{aligned} f(u(t), v(t), w(t), x(t)) &= (v(t) + w(t))^2 - 4u(t)x(t) \\ g(u(t), v(t), w(t), x(t)) &= v(t)\sin^2(\frac{\pi}{2}t) - w(t)\cos^2(\frac{\pi}{2}t) \end{aligned} \tag{38}$$

From these conditions, we obtain

$$v(t) = 2\sqrt{u(t)}\sqrt{x(t)}\cos^2(\frac{\pi}{2}t) \tag{39}$$
$$w(t) = 2\sqrt{u(t)}\sqrt{x(t)}\sin^2(\frac{\pi}{2}t) \tag{40}$$

Three types of curve segments, i.e. parabola, elliptic arc and hyperbola, can be uniformly represented by

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rational Bézier curves and a curve segment of C^2 interpolating splines can be expressed

$$u_0(t) = \frac{(1-t)^2}{r_0(t)}\cos^2(\frac{\pi}{2}t) + \frac{(1-t)^2}{r_1(t)}\sin^2(\frac{\pi}{2}t) = (1-t)^2(\frac{1}{r_0(t)}\cos^2(\frac{\pi}{2}t) + \frac{1}{r_1(t)}\sin^2(\frac{\pi}{2}t)) \tag{41}$$

$$v_0(t) = \frac{2(1-t)t\,w_1}{r_0(t)}\cos^2(\frac{\pi}{2}t) \tag{42}$$

$$w_0(t) = \frac{2(1-t)t\,w_2}{r_1(t)}\sin^2(\frac{\pi}{2}t) \tag{43}$$

$$x_0(t) = \frac{t^2}{r_0(t)}\cos^2(\frac{\pi}{2}t) + \frac{t^2}{r_1(t)}\sin^2(\frac{\pi}{2}t) = t^2(\frac{1}{r_0(t)}\cos^2(\frac{\pi}{2}t) + \frac{1}{r_1(t)}\sin^2(\frac{\pi}{2}t)) \tag{44}$$

where $r_0(t) = (1-t)^2 + 2(1-t)t\,w_1 + t^2$, $r_1(t) = (1-t)^2 + 2(1-t)t\,w_2 + t^2$. Figure 4 shows blending functions with $w_1 = 0.5$ and $w_2 = 1$. Hence

$$u(t) + v(t) + w(t) + x(t) = 1 \tag{45}$$

$$(\frac{v(t)}{w_1} + \frac{w(t)}{w_2})^2 = 4u(t)x(t) \tag{46}$$

$$\frac{v(t)r_0(t)}{w_1}\sin^2(\frac{\pi}{2}t) = \frac{v(t)r_1(t)}{w_2}\cos^2(\frac{\pi}{2}t) \tag{47}$$

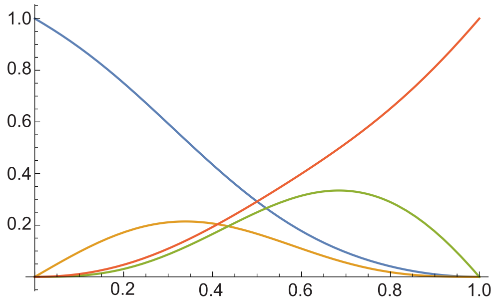

Figure 4: Blending functions of C^2 interpolating spline of Bézier type.

5.3 Rationalization of C^2 Interpolating Spline

Type 2 of circular arc and type 3 of elliptic arc can be represented by rational quadratic Bézier curves. As discussed in the previous subsection, we would like to specify arbitrary weight for the second control point as a generalization of C^2 interpolating spline. For curve generation, at first we give a weight for each input point. A curve segment is determined by three control points and three weights. We perform normalization of the weights as follows:

$$w'_0 = 1 \tag{48}$$

$$w'_1 = \frac{w_1}{\sqrt{w_0 w_2}} \tag{49}$$

$$w'_2 = 1 \tag{50}$$

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Note that the shape of the curve remains the same, but the location of the point on the curve for a given parameter value is generally different from the original position. Hence, the position of the point made by the blending of $\{\cos^2(\pi t/2), \sin^2(\pi t/2)\}$ is generally different. Therefore, for the shape calculation, the normalization of the weight is performed first before the blending is conducted.

Figure 5 shows various curve examples for given weights.

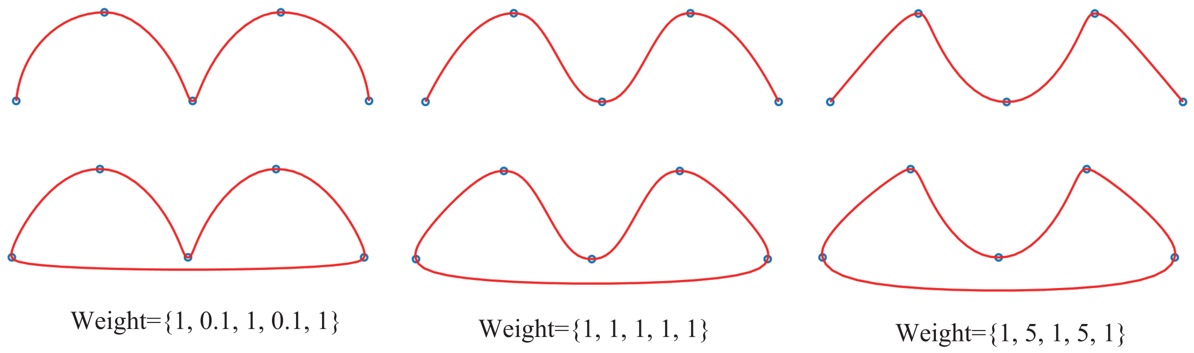

Figure 5: Rational C^2 interpolating spline.

5.4 Polynomial Blending Bézier type C^2 Interpolating Splines

We change $\{\cos^2(\pi t/2), \sin^2(\pi t/2)\}$ to

$$u(t) = B_0^3(t) + B_1^3(t) = (1-t)^2(1+2t) \tag{51}$$

$$v(t) = B_2^3(t) + B_3^3(t) = (3-2t)t^2 \tag{52}$$

where $B_i^3(t) = \frac{3!}{(3-i)!i!}(1-t)^{(3-i)}t^i$. Figure 6(a) shows these blending functions. Furthermore Figure 6(b) shows graphs of $\{\cos^2(\pi t/2), \sin^2(\pi t/2)\}$ with these blending functions. They are almost the same. By using

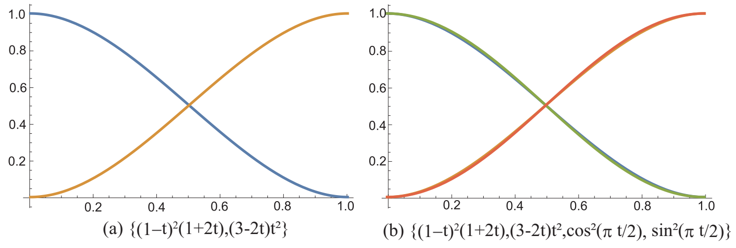

Figure 6: Blending functions.

these blending functions, we can represent C^2 interpolating splines with B-spline curves. Notice that from the shape uniqueness theorem, strictly speaking their shapes are generally different. However, the differences are quite small as shown in Fig.7.

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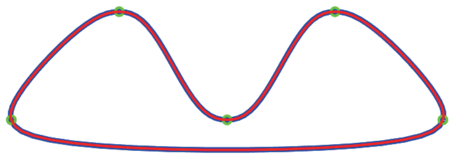

Figure 7: Polynomial blending Bézier type C^2 Interpolating spline curve in red with the original curve in blue.

5.5 Shape Uniqueness Theorem for General Degrees

By replacing Eq.(24) as follows:

$$\begin{aligned} f_1(b_0, b_1, \cdots, b_n) &= 0 \\ f_2(b_0, b_1, \cdots, b_n) &= 0 \\ &\cdots \\ f_{n-1}(b_0, b_1, \cdots, b_n) &= 0 \end{aligned} \quad (53)$$

where f_i do not depend on parameter t, we can extend the shape uniqueness theorem for the general degree n.

6 CONCLUSIONS

In this study, we consider the cases where the blending functions satisfy Gobithaasan-Miura's recursive algorithm [3] and those where they do not. A higher-order (including third-order) version of the shape uniqueness theorem has been presented. The blending functions of one curve segment of C^2 interpolating spline of quadratic Bézier curve type has been derived and we have shown that they satisfy the shape uniqueness theorem of the case where Gobithaasan-Miura's recursive algorithm is not satisfied. In the future, we will perform further improvement of the theorem to include more various cases.

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