# Subdivision Depth Computation for Catmull-Clark Subdivision Surfaces

Fuhua (Frank) Cheng[1] and Jun-Hai Yong[2]

[1]University of Kentucky, cheng@cs.uky.edu
[2]Tsinghua University, yongjh@tsinghua.edu.cn

## ABSTRACT

A subdivision depth computation technique for Catmull-Clark subdivision surfaces (CCSS's) is presented. The subdivision depth computation technique also includes distance evaluation techniques for CCSS patches with their control meshes. The distance and the subdivision depth computation techniques provide the long-needed precision/error control tools in subdivision surface trimming, finite element mesh generation, Boolean operations, and surface tessellation for rendering processes.

**Keywords:** subdivision surfaces, distance evaluation, subdivision depth computation.

## 1. INTRODUCTION

Subdivision surfaces have become popular recently in graphical modeling, animation and CAD/CAM because of their stability in numerical computation, simplicity in coding and, most importantly, their capability in modeling/representing complex shape of arbitrary topology. Given a control mesh and a set of *mesh refining rules* (or, more intuitively, *corner cutting rules*), one gets a *limit surface* by recursively cutting off corners of the control mesh [3, 6]. The limit surface is called a *subdivision surface* because the corner cutting (mesh refining) process is a generalization of the uniform B-spline surface *subdivision technique*. Subdivision surfaces include uniform B-spline surfaces and piecewise Bezier surfaces as special cases. Actually subdivision surfaces include non-uniform B-spline surfaces and NURBS surfaces as special cases as well [11]. Subdivision surfaces can model/represent complex shape of arbitrary topology because there is no limit on the shape and topology of the control mesh of a subdivision surface. With the parametrization technique of subdivision surfaces becoming available [12], we now know that subdivision surfaces cover both *parametric forms* and *discrete forms*. Since parametric forms are good for design and representation and discrete forms are good for machining and tessellation (including FE mesh generation) [1], we finally have a representation scheme that is good for all graphics and CAD/CAM applications.

Research work for subdivision surfaces has been done in several important areas, such as surface trimming [8], Boolean operations [2], and mesh editing [14]. However, the area of *precision/error control* for Catmull-Clark subdivision surfaces (CCSS's) is almost blank. For instance, given an error tolerance, how many levels of recursive Catmull-Clark subdivision should be performed on the initial control mesh so that the distance between the resultant control mesh and the limit surface would be less than the error tolerance? This error control technique is required in all tessellation based applications such as subdivision surface trimming, finite element mesh generation, Boolean operations, and surface tessellation for rendering. A subdivision depth computation method based on an exponential bound of the distance between a CCSS patch and its control mesh has been presented [13]. Unfortunately, some of the claims in this paper do not seem to be true. For instance, the distance between a CCSS patch and its control mesh need not reach the maximum at a control vertex. A subdivision depth computation technique based on bounds of second derivatives has been presented for tensor product rational surfaces [4]. The technique used for tensor product rational surfaces can not be used here because the parameter space of a CCSS usually does not fit into a rectangular grid structure.

In this paper we present a different technique to compute the subdivision depth for a CCSS patch. The subdivision depth computation technique also includes distance evaluation techniques for a CCSS patch with its control mesh. The new techniques are based on the control points of the CCSS patch only and work for CCSS patches with or without an extraordinary vertex. The presented subdivision depth computation technique provides an efficient error control tool that works for all tessellation based applications of CCSS's. A potential disadvantage of the subdivision depth computation technique is that it might generate a relatively large subdivision depth for a patch with an extraordinary vertex even though the patch is already flat enough. This is due to the fact that the first order norm can

not measure the curvature difference between two points. A possible solution to this problem in given in the last section.

## 2. SUBDIVISION DEPTH COMPUTATION FOR REGULAR PATCHES

Let $\mathbf{V}_0$, $\mathbf{V}_1$, $\mathbf{V}_2$ and $\mathbf{V}_3$ be the control points of a uniform cubic B-spline curve segment $\mathbf{C}(t)$ whose parameter space is [0, 1]. If we parametrize the middle leg of the control polygon as follows: $\mathbf{L}(t) = \mathbf{V}_1 + (\mathbf{V}_2 - \mathbf{V}_1)t$, $0 \le t \le 1$, then the maximum of $\|\mathbf{L}(t) - \mathbf{C}(t)\|$ is called the *distance* between the curve segment and its control polygon. It is *easy to see* that

$$\|\mathbf{L}(t) - \mathbf{C}(t)\| = \left\| \frac{(1-t)^3}{6}\left(2\mathbf{V}_1 - \mathbf{V}_0 - \mathbf{V}_2\right) + \frac{t^3}{6}\left(2\mathbf{V}_2 - \mathbf{V}_1 - \mathbf{V}_3\right) \right\| \le \frac{1}{6}\max\left\{\|2\mathbf{V}_1 - \mathbf{V}_0 - \mathbf{V}_2\|, \|2\mathbf{V}_2 - \mathbf{V}_1 - \mathbf{V}_2\|\right\}. \tag{1}$$

Since $(2\mathbf{V}_1 - \mathbf{V}_0 - \mathbf{V}_2)/6$ and $(2\mathbf{V}_2 - \mathbf{V}_1 - \mathbf{V}_3)/6$ are the values of $\mathbf{L}(t) - \mathbf{C}(t)$ at $t=0$ and $t=1$, we have the following lemma.

**Lemma 1:** The maximum of $\|\mathbf{L}(t) - \mathbf{C}(t)\|$ occurs at the endpoints of the curve segment and can be expressed as

$$\max_{t \in [0,1]} \|\mathbf{L}(t) - \mathbf{C}(t)\| = \frac{1}{6}\max\left\{\|2\mathbf{V}_1 - \mathbf{V}_0 - \mathbf{V}_2\|, \|2\mathbf{V}_2 - \mathbf{V}_1 - \mathbf{V}_3\|\right\}. \tag{2}$$

A form more general than Eqn. (1) has been proved by Peters [9]. His result works for uniform B-spline curves of any degree. However, the above result is more intuitive and is all we need for subsequent results. We next define the *distance* between a uniform bicubic B-spline surface patch and its control mesh.

Let $\mathbf{V}_{i,j}$, $0 \le i, j \le 3$, be the control points of a uniform bicubic B-spline surface patch $\mathbf{S}(u,v)$ with parameter space $[0,1] \times [0,1]$. If we parametrize the central mesh face $\{ \mathbf{V}_{1,1}, \mathbf{V}_{2,1}, \mathbf{V}_{1,2}, \mathbf{V}_{2,2} \}$ as follows:
$$\mathbf{L}(u,v) = (1-v)[(1-u)\mathbf{V}_{1,1} + u\mathbf{V}_{2,1}] + v[(1-u)\mathbf{V}_{1,2} + u\mathbf{V}_{2,2}], \quad 0 \le u, v \le 1,$$
then the maximum of $\|\mathbf{L}(u,v) - \mathbf{S}(u,v)\|$ is called the *distance* between $\mathbf{S}(u,v)$ and its control mesh. If we define

$$\mathbf{Q}_{u,k} \equiv (1-u)\mathbf{V}_{1,k} + u\mathbf{V}_{2,k}, \quad \mathbf{Q}_{v,k} \equiv (1-v)\mathbf{V}_{k,1} + v\mathbf{V}_{k,2}, \quad \overline{\mathbf{Q}}_{u,k} \equiv \sum_{i=0}^{3} N_{i,3}(u)\mathbf{V}_{i,k} \quad \text{and} \quad \overline{\mathbf{Q}}_{v,k} \equiv \sum_{j=0}^{3} N_{j,3}(u)\mathbf{V}_{k,j}, \quad \text{where } N_{i,3}(t) \text{ are}$$

standard uniform B-spline basis functions of degree three, we have

$$\|\mathbf{L}(u,v) - \mathbf{S}(u,v)\| \le (1-v)\left\|\mathbf{Q}_{u,1} - \overline{\mathbf{Q}}_{u,1}\right\| + v\left\|\mathbf{Q}_{u,2} - \overline{\mathbf{Q}}_{u,2}\right\| + \sum_{i=0}^{3} N_{i,3}(u)\left\|\mathbf{Q}_{v,i} - \overline{\mathbf{Q}}_{v,i}\right\|.$$

By applying Lemma 1 on $\|\mathbf{Q}_{u,1} - \overline{\mathbf{Q}}_{u,1}\|$, $\|\mathbf{Q}_{u,2} - \overline{\mathbf{Q}}_{u,2}\|$, and $\|\mathbf{Q}_{v,i} - \overline{\mathbf{Q}}_{v,i}\|$, $i = 1, 2, 3$, and by defining $M^0$ as the maximum norm of the second order forward differences of the control points of $\mathbf{S}(u,v)$, we have

$$\|\mathbf{L}(u,v) - \mathbf{S}(u,v)\| \le \frac{1}{6}\left[(1-v)M^0 + vM^0 + \sum_{i=0}^{3} N_{i,3}(u)M^0\right] \le \frac{1}{3}M^0.$$

$M^0$ is called the *second order norm* of $\mathbf{S}(u,v)$. This leads to the following lemma.

**Lemma 2:** The maximum of $\|\mathbf{L}(u,v) - \mathbf{S}(u,v)\|$ satisfies the following inequality

$$\max_{0 \le u,v \le 1} \|\mathbf{L}(u,v) - \mathbf{S}(u,v)\| \le \frac{1}{3}M^0, \tag{3}$$

where $M^0$ is the second order norm of $\mathbf{S}(u,v)$.

Note that even though the maximum of $\|\mathbf{L}(t) - \mathbf{C}(t)\|$ occurs at the end points of the curve segment $\mathbf{C}(t)$, the maximum of $\|\mathbf{L}(u,v) - \mathbf{S}(u,v)\|$ for a surface patch usually does not occur at the corners of $\mathbf{S}(u,v)$. This follows from the observation that if this is the case then the maximum of $\|\mathbf{L}(u,0) - \mathbf{S}(u,0)\|$ would occur at the endpoints of $\mathbf{S}(u,0)$. But unlike the curve case, $\|\mathbf{L}(u,0) - \mathbf{S}(u,0)\|$ does not satisfy an expression of the following form
$$(1-t)^3(2\mathbf{V}_1 - \mathbf{V}_0 - \mathbf{V}_2)/6 + t^3(2\mathbf{V}_2 - \mathbf{V}_1 - \mathbf{V}_3)/6$$
What we have is the one shown below
$$\mathbf{L}(u,0) - \mathbf{S}(u,0) = (1-u)^3[\mathbf{L}(0,0) - \mathbf{S}(0,0)] + u^3[\mathbf{L}(1,0) - \mathbf{S}(1,0)] + u(1-u)(2-u)[\mathbf{L}(0,0) - (\mathbf{V}_{1,0} + 2\mathbf{V}_{1,1} + \mathbf{V}_{1,2})/6]$$
$$+ u(1-u)(2-u)[\mathbf{L}(1,0) - (\mathbf{V}_{2,0} + 2\mathbf{V}_{2,1} + \mathbf{V}_{2,2})/6].$$

The second and third terms on the right hand side usually are non-zero (actually, they could be very large in magnitude if $\mathbf{V}_{1,0}$ and $\mathbf{V}_{2,0}$ are very far away from the other control points). Therefore, the maximum of $\|\mathbf{L}(u,0) - \mathbf{S}(u,0)\|$ is not necessary to occur at the endpoints of $\mathbf{S}(u,0)$. Consequently, the maximum of $\|\mathbf{L}(u,v) - \mathbf{S}(u,v)\|$ is not necessary to occur at the corners of $\mathbf{S}(u,v)$. In the following, we present subdivision depth computation technique for CCSS patches not adjacent to an extraordinary vertex.

Let $\mathbf{V}_{i,j}$, $0 \le i, j \le 3$, be the control points of a uniform bicubic B-spline surface patch $\mathbf{S}(u,v)$. We use $\mathbf{V}_{i,j}{}^{k}$, $0 \le i, j \le 3+2^{k}-1$, to represent the new control points of the surface patch after $k$ levels of recursive subdivision. The indexing of the new control points follows the convention that $\mathbf{V}_{0,0}{}^{k}$ is always the *face point* of the mesh face { $\mathbf{V}_{0,0}{}^{k-1}$, $\mathbf{V}_{1,0}{}^{k-1}$, $\mathbf{V}_{0,1}{}^{k-1}$, $\mathbf{V}_{1,1}{}^{k-1}$ }. The new control points $\mathbf{V}_{i,j}{}^{k}$ will be called the *level-k control points* of $\mathbf{S}(u,v)$ and the new control mesh will be called the *level-k control mesh* of $\mathbf{S}(u,v)$.

Note that if we divide the parameter space of the surface patch into $4^{k}$ regions as follows:

$$\boldsymbol{\Omega}_{m,n}^{k} = \left[\frac{m}{2^k}, \frac{m+1}{2^k}\right] \times \left[\frac{n}{2^k}, \frac{n+1}{2^k}\right], \tag{4}$$

where $0 \le m, n \le 2^{k}-1$ and let the corresponding subpatches be denoted $\mathbf{S}_{m,n}{}^{k}(u,v)$, then each $\mathbf{S}_{m,n}{}^{k}(u,v)$ is a uniform bicubic B-spline surface patch defined by the level-$k$ control point set { $\mathbf{V}_{p,q}{}^{k} \mid m \le p \le m+3$, $n \le q \le n+3$ }. $\mathbf{S}_{m,n}{}^{k}(u,v)$ is called a *level-k subpatch* of $\mathbf{S}(u,v)$. One can define a level-$k$ bilinear plane $\mathbf{L}_{m,n}{}^{k}$ on { $\mathbf{V}_{p,q}{}^{k} \mid p = m+1, m+2; q = n+1, n+2$ } and measure the distance between $\mathbf{L}_{m,n}{}^{k}(u,v)$ and $\mathbf{S}_{m,n}{}^{k}(u,v)$. We say that the *distance between* $\mathbf{S}(u,v)$ *and the level-k control mesh is smaller than*} $\varepsilon$ if the distance between each level-$k$ subpatch $\mathbf{S}_{m,n}{}^{k}(u,v)$ and the corresponding level-$k$ bilinear plane $\mathbf{L}_{m,n}{}^{k}(u,v)$, $0 \le m, n \le 2^{k}-1$, is smaller than $\varepsilon$. In the following, we will show how to compute a subdivision depth $k$ for a given $\varepsilon$ so that the distance between $\mathbf{S}(u,v)$ and the level-$k$ control mesh is smaller than $\varepsilon$ after $k$ levels of recursive subdivision. The following lemma is needed in the derivation of the computation process. If we use $M_{m,n}{}^{k}$ to represent the second order norm of $\mathbf{S}_{m,n}{}^{k}(u,v)$, i.e., the maximum norm of the second order forward differences of the control points of $\mathbf{S}_{m,n}{}^{k}(u,v)$, then the lemma shows the second order norm of $\mathbf{S}_{m,n}{}^{k}(u,v)$ converges at a rate of 1/4 of the level-($k$-1) second order norm. The proof of this lemma is shown in the complete version of the paper [5].

**Lemma 3:** If $M_{m,n}{}^{k}$ is the second order norm of $\mathbf{S}_{m,n}{}^{k}(u,v)$ then we have

$$M_{m,n}^{k} \le \left(\frac{1}{4}\right)^{k} M^{0}, \tag{5}$$

where $M^{0}$ is the second order norm of $\mathbf{S}(u,v)$.

With Lemmas 2 and 3, it is easy to see that, for any $0 \le m, n \le 2^{k-1}$, we have

$$\max_{0 \le u,v \le 1} \left\| \mathbf{L}_{m,n}^{k}(u,v) - \mathbf{S}_{m,n}^{k}(u,v) \right\| \le \frac{1}{3} M_{m,n}^{k} \le \frac{1}{3} \times \left(\frac{1}{4}\right)^{k} M^{0}. \tag{6}$$

Hence, if $k$ is large enough to make the right side of Eqn. (6) smaller than $\varepsilon$, we have

$$\max_{0 \le u,v \le 1} \left\| L_{m,n}^{k}(u,v) - S_{m,n}^{k}(u,v) \right\| \le \varepsilon$$

for every $0 \le m, n \le 2^{k-1}$. This leads to the following main result of this subsection.

**Theorem 4:** Let $\mathbf{V}_{i,j}$, $0 \le i, j \le 3$, be the control points of a uniform bicubic B-spline surface patch $\mathbf{S}(u,v)$. For any given $\varepsilon > 0$, if

$$k \ge \left\lceil \log_4 \left(\frac{M^0}{3\varepsilon}\right) \right\rceil \tag{7}$$

levels of recursive subdivision are performed on the control points of $\mathbf{S}(u,v)$ then the distance between $\mathbf{S}(u,v)$ and the level-$k$ control mesh is smaller than $\varepsilon$ where $M^{0}$ is the second order norm of $\mathbf{S}(u,v)$.

## 3. SUBDIVISION DEPTH COMPUTATION FOR EXTRA-ORDINARY PATCHES

The subdivision depth computation process for a CCSS patch near an extraordinary vertex is different. This is because in the vicinity of an extraordinary vertex one does not have a uniform B-spline surface patch representation and, consequently, cannot use the technique of Theorem 4 directly. Fortunately, the size of such a vicinity can be made as small as possible, therefore, one can reduce the size of such a vicinity to a degree that is tolerable (i.e., within the given

error bound) and use the technique of Theorem 4 to work on the remaining part of the surface patch. A subdivision depth computation technique based on this concept for a CCSS patch near an extraordinary vertex will be presented below. We assume the initial mesh has been subdivided at least twice so that each mesh face is a quadrilateral and contains at most one extraordinary vertex. We need to define a few notations first.

Let $\Pi_0^0 = \{ \mathbf{V}_i \mid 1 \leq i \leq 2N+8 \}$ be a level-0 control point set that influences the shape of a surface patch $\mathbf{S}(u,v)$ ($=\mathbf{S}_0^0(u,v)$). $\mathbf{V}_1$ is an *extraordinary vertex* with *valence N*. The control vertices are ordered following Stam's fashion [12] (see Fig. 1.).
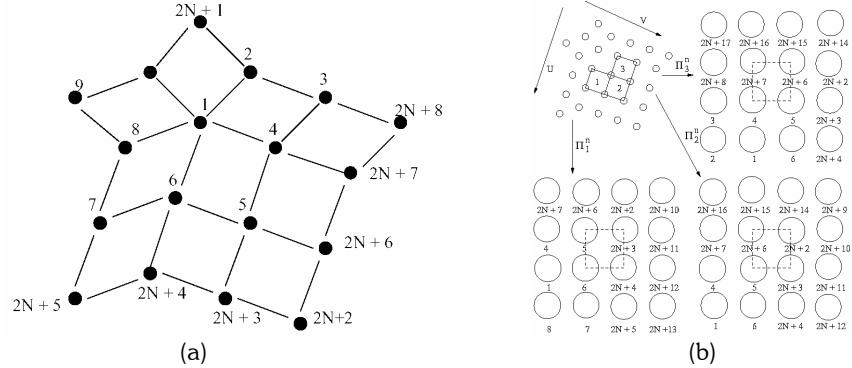


Fig. 1. (a) Ordering of control points for an extra-ordinary CCSS patch; (b) Control point sets $\Pi_1^n$, $\Pi_2^n$ and $\Pi_3^n$.

If we use $\mathbf{V}_i^n$ to represent the level-$n$ control vertices generated after $n$ levels of recursive Catmull-Clark subdivision, and use $\mathbf{S}_0^n$, $\mathbf{S}_1^n$, $\mathbf{S}_2^n$ and $\mathbf{S}_3^n$ to represent the subpatches of $\mathbf{S}_0^{n-1}$ defined over the tiles

$$\mathbf{\Omega}_0^n = \left[0, \frac{1}{2^n}\right] \times \left[0, \frac{1}{2^n}\right], \ \mathbf{\Omega}_1^n = \left[\frac{1}{2^n}, \frac{1}{2^{n-1}}\right] \times \left[0, \frac{1}{2^n}\right], \ \mathbf{\Omega}_2^n = \left[\frac{1}{2^n}, \frac{1}{2^{n-1}}\right] \times \left[\frac{1}{2^n}, \frac{1}{2^{n-1}}\right],$$

$\mathbf{\Omega}_3^n = \left[0, \frac{1}{2^n}\right] \times \left[\frac{1}{2^n}, \frac{1}{2^{n-1}}\right]$, respectively, then the shape of $\mathbf{S}_0^n$, $\mathbf{S}_1^n$, $\mathbf{S}_2^n$ and $\mathbf{S}_3^n$ are influenced by the level-$n$ control point sets $\Pi_0^n$, $\Pi_1^n$, $\Pi_2^n$ and $\Pi_3^n$, respectively. $\Pi_0^n$ is defined by $\Pi_0^n = \{ \mathbf{V}_i^n \mid 1 \leq i \leq 2N+8 \}$, and definition of $\Pi_1^n$, $\Pi_2^n$ and $\Pi_3^n$ can be found in Fig. 1(b). $\mathbf{S}_1^n$, $\mathbf{S}_2^n$ and $\mathbf{S}_3^n$ are standard uniform bicubic B-spline surface patches because their control meshes satisfy a 4-by-4 structure. Hence, the technique described in Theorem 4 can be used to compute a subdivision depth for each of them. $\mathbf{S}_0^n$ is not a standard uniform bicubic B-spline surface patch. Hence, Theorem 4 can not be used to compute a subdivision depth for $\mathbf{S}_0^n$ directly. For the convenience of reference, we shall call $\mathbf{S}_0^n$ a *level-n extraordinary subpatch* of $\mathbf{S}(u,v)$ because it contains the limit point of the extraordinary points (The proof is in the next subsection.) Note that if $\mathbf{H}_0$ and $\mathbf{H}_n$ are column vector representations of the control points of $\Pi_0^0$ and $\Pi_0^n$, respectively, $\mathbf{H}_0 \equiv (\mathbf{V}_0, \mathbf{V}_1, ..., \mathbf{V}_{2N+8})^t$, and $\mathbf{H}_n \equiv (\mathbf{V}_0^n, \mathbf{V}_1^n, ..., \mathbf{V}_{2N+8}^n)^t$, where $(\mathbf{X}, \mathbf{X}, ..., \mathbf{X})^t$ represents the transpose of the row vector $(\mathbf{X}, \mathbf{X}, ..., \mathbf{X})$ then we have

$$\mathbf{H}_n = (\mathbf{T})^n \mathbf{H}_0, \tag{8}$$

where $\mathbf{T}$ is the $(2N+8) \times (2N+8)$ (extended) subdivision matrix defined as follows [7, 12]:

$$T = \begin{pmatrix} \overline{T} & 0 \\ \overline{T}_{1,1} & \overline{T}_{1,2} \end{pmatrix}, \tag{9}$$

with

$$\overline{T} = \begin{pmatrix} a_N & b_N & c_N & b_N & c_N & b_N & \cdots & b_N & c_N \\ d & d & e & e & 0 & 0 & \cdots & e & e \\ f & f & f & f & 0 & 0 & \cdots & 0 & 0 \\ d & e & e & d & e & e & \cdots & 0 & 0 \\ f & 0 & 0 & f & f & f & \cdots & 0 & 0 \\ & & \vdots & & & & \ddots & \vdots & \\ d & e & 0 & 0 & 0 & 0 & \cdots & d & e \\ f & f & 0 & 0 & 0 & 0 & \cdots & f & f \end{pmatrix}, \tag{10}$$

$$\overline{T}_{1,1} = \begin{pmatrix} c & 0 & 0 & b & a & b & 0 & 0 & 0 \\ e & 0 & 0 & e & d & d & 0 & 0 & 0 \\ b & 0 & 0 & c & b & a & b & c & 0 \\ e & 0 & 0 & 0 & 0 & d & d & e & 0 \\ e & 0 & 0 & d & d & e & 0 & 0 & 0 \\ b & c & b & a & b & c & 0 & 0 & 0 \\ e & e & d & d & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$ (11)

$$\overline{T}_{1,2} = \begin{pmatrix} c & b & c & 0 & b & c & 0 \\ 0 & e & e & 0 & 0 & 0 & 0 \\ 0 & c & b & c & 0 & 0 & 0 \\ 0 & 0 & e & e & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e & e & 0 \\ 0 & 0 & 0 & 0 & c & b & c \\ 0 & 0 & 0 & 0 & 0 & e & e \end{pmatrix},$$ (12)

and $a_N = 1 - 7/(4N)$, $b_N = 3/(2N^2)$, $c_N = 1/(4N^2)$, $a = 9/16$, $b = 3/32$, $c = 1/64$, $d = 3/8$, $e = 1/16$, $f = 1/4$.

### 3.1 Computing Subdivision Depth for a Vicinity of the Extraordinary Vertex

The goal here is to find an integer $n_\varepsilon$ for a given $\varepsilon > 0$ so that if $n$ ($\geq n_\varepsilon$) recursive subdivisions are performed on $\Pi_0^0$, then the control point set of the level-$n$ extraordinary subpatch $\mathbf{S}_0^n$ of $\mathbf{S}(u,v)$, $\Pi_0^n = \{ \mathbf{V}_i^n \mid 1 \leq i \leq 2N+8 \}$, is contained in the sphere $\mathbf{B}(\mathbf{V}_5^{n+1}, \varepsilon/2)$ with center $\mathbf{V}_5^{n+1} \equiv (\mathbf{V}_1^n + \mathbf{V}_4^n + \mathbf{V}_5^n + \mathbf{V}_6^n)/4$ and radius $\varepsilon/2$. Note that if the $(2N+8)$-point control mesh $\Pi_0^n$ is contained in the sphere $\mathbf{B}(\mathbf{V}_5^{n+1}, \varepsilon/2)$ then the level-$n$ extraordinary subpatch $\mathbf{S}_0^n$ is contained in the sphere $\mathbf{B}(\mathbf{V}_5^{n+1}, \varepsilon/2)$ as well. This follows from the fact that $\mathbf{S}_0^n$, as the limit surface of $\Pi_0^n$, is contained in the *convex hull* of $\Pi_0^n$ and the convex hull of $\Pi_0^n$ is contained in the sphere $\mathbf{B}(\mathbf{V}_5^{n+1}, \varepsilon/2)$. But then we have

$$\max \ \| \ \mathbf{S}_0^n(u,v) - \mathbf{L}_0^n(u,v) \ \| \ < \ \varepsilon$$ (13)

where $\mathbf{L}_0^n(u,v)$ is a bilinear plane defined on the level-$n$ mesh face $\{\mathbf{V}_1^n, \mathbf{V}_4^n, \mathbf{V}_5^n, \mathbf{V}_6^n \}$. The construction of such an $n_\varepsilon$ depends on several properties of the (extended) subdivision matrix $\mathbf{T}$ and the control point sets $\Pi_0^n$.

First note that since all the entries of the extended subdivision matrix $\mathbf{T}$ are non-negative and the sum of each row equals one, the extended subdivision matrix is a *transition probability matrix* of a $(2N+8)$-state Markov chain [10]. In particular, the $(2N+1) \times (2N+1)$ block $\overline{T}$ of $\mathbf{T}$ is a *transition probability matrix* of a $(2N+1)$-state Markov chain. The entries in the first row and first column of $\overline{T}$ are all non-zero. Therefore, the matrix $\overline{T}$ is *irreducible* because $\overline{T}^2$ has no zero entries and, consequently, all the states are accessible to each other. On the other hand, since all the diagonal entries of $\overline{T}$ are non-zero and entries of $\overline{T}^n$ are non-zero for all $n \geq 2$, it follows that all the states of $\overline{T}$ are *aperiodic* and *positive recurrent*. Consequently, the Markov chain is *irreducible* and *ergodic*. By the well-known theorem of Markov chain ([10], Theorem 4.1), $\overline{T}^n$ converges to a limit matrix $\overline{T}^*$ whose rows are identical. More precisely,

$$\lim_{n \to \infty} \overline{T}^n = \overline{T}^* \equiv \begin{pmatrix} \Delta_1 & \Delta_2 & \cdots & \Delta_{2N+1} \\ \Delta_1 & \Delta_2 & \cdots & \Delta_{2N+1} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_1 & \Delta_2 & \cdots & \Delta_{2N+1} \end{pmatrix},$$ (14)

where $\Delta_i$ are the unique non-negative solution of

$$\begin{cases} \Delta_j = \sum_{i=1}^{2N+1} \Delta_i \overline{t}_{i,j}, & j = 1, 2, \cdots 2N+1, \\ \sum_{j=1}^{2N+1} \Delta_j = 1, \end{cases}$$ (15)

with $\overline{t}_{i,j}$ being the entries of $\overline{T}$. One can easily get the following observations.

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- The vector ( $\Delta_1$, $\Delta_2$, ..., $\Delta_{2N+1}$) satisfies the following properties:
  $\Delta_1 = N/(N+5)$, $\Delta_2 = \Delta_4 = ... = \Delta_{2N} = 4/(N(N+5))$, and $\Delta_3 = \Delta_5 = ... = \Delta_{2N+1} = 1/(N(N+5))$.

- The matrix $\overline{T}^*$ is an idempotent matrix, i.e., $\overline{T}^* \overline{T}^* = \overline{T}^*$. Hence, $\overline{T}^*$ has two eigenvalues, 1 and 0 (with multiplicity $2N$).

- $\overline{T}$ has 1 as an eigenvalue and all the other $2N$ eigenvalues of $\overline{T}$ have a magnitude smaller than one.

- As it is well known [7], the limit point of $\{ \mathbf{V}_1^n \}$ is $\mathbf{V}_1^* \equiv \Delta_1 \mathbf{V}_1 + \Delta_2 \mathbf{V}_2 + ... + \Delta_{2N+1} \mathbf{V}_{2N+1}$. But $\mathbf{V}_1^*$ is actually the limit point of all $\mathbf{V}_j^n$, $j=1, 2, ... , 2N+8$. Therefore, the convex hull of $\{ \mathbf{V}_1^n, \mathbf{V}_2^n, ... , \mathbf{V}_{2N+8}^n \}$ converges to $\mathbf{V}_1^*$ when $n$ tends to infinity and, consequently, $\mathbf{V}_1^* = \mathbf{S}(0,0)$. The fact that $\mathbf{V}_1^*$ is the limit point of $\{ \mathbf{V}_1^n, \mathbf{V}_2^n, ... , \mathbf{V}_{2N+1}^n \}$ follows from Eqns.(8) and (14). The fact that $\mathbf{V}_1^*$ is also the limit point of $\{ \mathbf{V}_{2N+2}^n, \mathbf{V}_{2N+3}^n, ... , \mathbf{V}_{2N+8}^n \}$ is proved in the complete version of the paper [5].

The last observation is important because it shows that

$$\max_{\mathbf{V} \in \Pi_0^n} \left\| \mathbf{V}_5^{n+1} - \mathbf{V} \right\| \tag{16}$$

converges. Therefore, it is possible to reduce the size of $\mathbf{S}_0^n$ to a degree that is tolerable if $n$ is large enough. For a given $\varepsilon > 0$ we will find an $n_\varepsilon$ so that if $n \geq n_\varepsilon$ then the level-$n$ control point set $\Pi_0^n$ is contained in the sphere $\mathbf{B}(\mathbf{V}_5^{n+1}, \varepsilon/2)$. To do this, we need to know how fast Eqn. (16) converges.
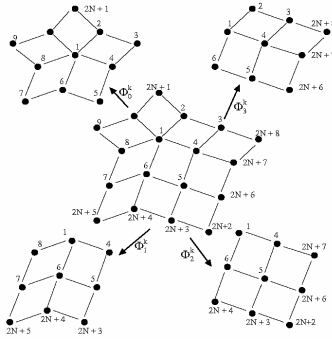


Fig. 2. Control point sets $\mathbf{\Phi}_0^n$, $\mathbf{\Phi}_1^n$, $\mathbf{\Phi}_2^n$ and $\mathbf{\Phi}_3^n$.

Let $\mathbf{\Phi}_0^k$, $\mathbf{\Phi}_1^k$, $\mathbf{\Phi}_2^k$ and $\mathbf{\Phi}_3^k$ be subsets of $\Pi_0^k$ defined as follows (see Fig. 2.):

$$\begin{cases} \Phi_0^k = \left\{ \mathbf{V}_j^k \mid j = 1,2,\cdots,2N+1 \right\}, \\ \Phi_1^k = \left\{ \mathbf{V}_j^k \mid j = 1,4,5,\cdots 8,2N+3,2N+4,2N+5 \right\}, \\ \Phi_2^k = \left\{ \mathbf{V}_j^k \mid j = 1,4,5,6,2N+2,2N+3,2N+4,2N+6,2N+7 \right\}, \\ \Phi_3^k = \left\{ \mathbf{V}_j^k \mid j = 1,2,\cdots,6,2N+6,2N+7,2N+8 \right\}, \end{cases} \tag{17}$$

($\mathbf{V}_8^k$ in $\mathbf{\Phi}_1^k$ should be replaced with $\mathbf{V}_2^k$ if $N=3$) and define $G_0^k$, $G_1^k$, $G_2^k$ and $G_3^k$ as follows:

$$\begin{cases} G_0^k = \max_{\mathbf{V} \in \Phi_0^k} \left\| \mathbf{V}_1^k - \mathbf{V} \right\|, \\ G_1^k = \max_{\mathbf{V} \in \Phi_1^k} \left\| \mathbf{V}_6^k - \mathbf{V} \right\|, \\ G_2^k = \max_{\mathbf{V} \in \Phi_2^k} \left\| \mathbf{V}_5^k - \mathbf{V} \right\|, \\ G_3^k = \max_{\mathbf{V} \in \Phi_3^k} \left\| \mathbf{V}_4^k - \mathbf{V} \right\|. \end{cases} \tag{18}$$

$G_i^k$ is called the *first order norm* of $\mathbf{\Phi}_i^k$, $i=0, 1, 2, 3$. We need the following lemma for the construction of $n_\varepsilon$. The proof is shown in the complete version of the paper [5].

**Lemma 5:** If $\mathbf{\Phi}_i^k$ and $G_i^k$ are defined as above then, for $i=0, 1, 2, 3$, we have

$$G_i^k \leq \begin{cases} \left(\dfrac{3}{4}\right)^k G^0, & \text{if } N = 3, \\[3mm] \left(\dfrac{3}{4} + \dfrac{7}{4N} - \dfrac{13}{2N^2}\right)^k G^0, & \text{if } N \geq 5, \end{cases} \tag{19}$$

where $G^0 \equiv \max \{ G_0{}^0, G_1{}^0, G_2{}^0, G_3{}^0 \}$. $G_0$ is called the *first order norm* of $\Pi_0{}^0$.

To construct $n_\varepsilon$, note that if $\mathbf{V} \in \Pi_0{}^n$ and $\mathbf{V} \in \Phi_0{}^n$, we have

$$\left\|\mathbf{V}_5^{n+1} - \mathbf{V}\right\| \leq \frac{1}{4}\left\|\mathbf{V}_4^n - \mathbf{V}_1^n\right\| + \frac{1}{4}\left\|\mathbf{V}_5^n - \mathbf{V}_1^n\right\| + \frac{1}{4}\left\|\mathbf{V}_6^n - \mathbf{V}_1^n\right\| + \left\|\mathbf{V}_1^n - \mathbf{V}\right\| \leq \frac{7}{4}G_0^n .$$

It is easy to prove that similar inequalities hold for $\Phi_1{}^n$, $\Phi_2{}^n$ and $\Phi_3{}^n$ as well. Hence, for each $\mathbf{V} \in \Pi_0{}^n$, by Lemma 5, we have

$$\left\|\mathbf{V}_5^{n+1} - \mathbf{V}\right\| \leq \begin{cases} \dfrac{7}{4} \times \left(\dfrac{3}{4}\right)^n G^0, & \text{if } N = 3, \\[3mm] \dfrac{7}{4} \times \left(\dfrac{3}{4} + \dfrac{7}{4N} - \dfrac{13}{2N^2}\right)^n G^0, & \text{if } N \geq 5. \end{cases} \tag{20}$$

Since the maximum of $\dfrac{3}{4} + \dfrac{7}{4N} - \dfrac{13}{2N^2}$ occurs at $N{=}7$, Eqn. (20) can be simplified as

$$\left\|\mathbf{V}_5^{n+1} - \mathbf{V}\right\| \leq \frac{7}{4} \times \left(\frac{1}{\delta}\right)^n G^0, \tag{21}$$

where

$$\delta = \begin{cases} \dfrac{4}{3}, & \text{if } N = 3, \\[3mm] \dfrac{98}{85}, & \text{if } N \geq 5. \end{cases} \tag{22}$$

Hence, $\left\|\mathbf{V}_5^{n+1} - \mathbf{V}\right\|$ is smaller than $\varepsilon/2$ if $n$ is large enough to make the right hand side of Eqn. (21) smaller than or equal to $\varepsilon/2$. Consequently, we have the following theorem.

**Theorem 6:** Let $\Pi_0{}^0 = \{ \mathbf{V}_i \mid 1 \leq i \leq 2N+8 \}$ be a level-0 control point set that influences the shape of a CCSS patch $\mathbf{S}(u,v)$ $(=\mathbf{S}_0{}^0(u,v))$. $\mathbf{V}_1$ is an extraordinary vertex with *valence N*. The control vertices are ordered following Stam's fashion [12] (see Fig. 1.). For a given $\varepsilon > 0$, if $n_\varepsilon$ is defined as follows:

$$n_\varepsilon \equiv \left\lceil \log_\delta \left(\frac{7G^0}{2\varepsilon}\right) \right\rceil, \tag{23}$$

where $\delta$ is given in Eqn. (22) and $G^0$ is the first order norm of $\Pi_0{}^0$, then the distance between the level-$n$ extraordinary subpatch $\mathbf{S}_0{}^n(u,v)$ and the corresponding bilinear plane $\mathbf{L}_0{}^n(u,v)$ is smaller than or equal to $\varepsilon$ if $n \geq n_\varepsilon$.

Theorem 6 shows that the rate of convergence of the control mesh in the vicinity of an extraordinary vertex is fastest when valence of the extraordinary vertex is three.

### 3.2 Computing Subdivision Depth for the Remaining Part

The idea here is, for each $k$ between 1 and $n_\varepsilon$, to determine a subdivision depth $D_k$ ($\geq n_\varepsilon$) so that if $D_k$ recursive subdivisions are performed on the control mesh $\Pi_0{}^0$ of $\mathbf{S}(u,v)$, then the distance between the level-$D_k$ control mesh and the subpatches $\mathbf{S}_i^k$, $i{=}1, 2, 3$, is smaller than $\varepsilon$. Consequently, if we define $D$ to be the maximum of these $D_k$ (i.e., $D = \max \{ D_k \mid 1 \leq k \leq n_\varepsilon \}$), then after $D$ recursive subdivisions, the distance between the level-$D$ control mesh and the subpatches $\mathbf{S}_i^k$, $i{=}1, 2, 3$, would be smaller than $\varepsilon$ for all $1 \leq k \leq n_\varepsilon$. Note that the distance between the level-$D$ control mesh and the subpatches $\mathbf{S}_1{}^k$, $\mathbf{S}_2{}^k$ and $\mathbf{S}_3{}^k$ for $n_\varepsilon + 1 \leq k \leq D$, and the distance between the level-$D$ control mesh and the level-$D$ extraordinary subpatch $\mathbf{S}_0{}^D$ would be smaller than $\varepsilon$ as well. This is because these

subpatches are subpatches of $\mathbf{S}_0^{n_\varepsilon}$ and the distance between $\mathbf{S}_0^{n_\varepsilon}$ and the level-$n_\varepsilon$ control mesh is already smaller than $\varepsilon$. Hence, the key here is the construction of $D_k$. We will show the construction of $D_k$ for $\mathbf{S}_3^k(u,v)$. This $D_k$ works for $\mathbf{S}_1^k(u,v)$ and $\mathbf{S}_2^k(u,v)$ as well.

For $0 \leq u, v \leq 1$, define a bilinear plane $\mathbf{L}_3^k(u,v)$ on the mesh face $\{ \mathbf{V}_4^k, \mathbf{V}_5^k, \mathbf{V}_{2N+7}^k, \mathbf{V}_{2N+6}^k \}$ as follows:

$$\mathbf{L}_3^k(u,v) = (1-v)\,[\,(1-u)\,\mathbf{V}_4^k + u\,\mathbf{V}_5^k\,] + v\,[\,(1-u)\,\mathbf{V}_{2N+7}^k + u\,\mathbf{V}_{2N+6}^k\,]. \tag{24}$$

Since $\mathbf{S}_3^k(u,v)$ is a uniform bicubic B-spline surface patch with control mesh $\Pi_3^k(u,v)$, we have, by Lemma 2,

$$\left\| \mathbf{L}_3^k(u,v) - \mathbf{S}_3^k(u,v) \right\| \leq Z_3^k/3, \tag{25}$$

where $Z_3^k$ is the second order norm of $\mathbf{S}_3^k(u,v)$. If we define $Z_0^i$ to be the second order norm of $\mathbf{S}_0^i(u,v)$, we have

$$Z_3^k \leq W Z_0^{k-1} \leq (W)^k Z_0^0, \tag{26}$$

where

$$W = \begin{cases} \dfrac{2}{3}, & \text{if } N = 3, \\[2mm] \dfrac{1}{2} + \dfrac{1}{4N} + \dfrac{21}{4N^2}, & \text{if } N = 5, \\[2mm] \dfrac{3}{4} + \dfrac{2}{N} - \dfrac{21}{2N^2}, & \text{if } N > 5. \end{cases} \tag{27}$$

The proof of Eqn. (26) is shown in the complete version of the paper [5]. Hence, by combining the above results, we have

**Lemma 7:** The maximum distance between $\mathbf{S}_3^k$ and $\mathbf{L}_3^k$ satisfies the following inequality

$$\max \left\| \mathbf{L}_3^k(u,v) - \mathbf{S}_3^k(u,v) \right\| \leq (W)^k Z_0^0/3, \tag{28}$$

where $W$ is defined in Eqn. (27) and $Z_0^0$ is the second order norm of $\mathbf{S}(u,v)$.

It should be pointed out that when defining $Z_0^i$, only the following items are needed for second order forward differences involving $\mathbf{V}_1^i$:

$$\left\| 2\mathbf{V}_1^i - \mathbf{V}_{2j}^i - \mathbf{V}_{2[(j+2)\backslash\%N]}^i \right\|, \quad j = 1, 2, \ldots, N.$$

Lemma 7 shows that if $(W)^k Z_0^0/3 < \varepsilon$ then the distance between $\mathbf{S}_3^k$ and $\mathbf{L}_3^k$ is already smaller than $\varepsilon$. However, since $n_\varepsilon$ subdivisions have to be performed on $\Pi_0^0$ to get $\mathbf{S}_0^{n_\varepsilon}$ anyway, $D_k$ for $\mathbf{S}_3^k$ in this case is set to $n_\varepsilon$. This condition holds for $\mathbf{S}_1^k$ and $\mathbf{S}_2^k$ as well.

If $(W)^k Z_0^0/3 > \varepsilon$, further subdivisions are needed on $\Pi_i^k$, $i = 1, 2, 3$, to make the distance between $\mathbf{S}_i^k$, $i = 1, 2, 3$, and the corresponding mesh faces smaller than $\varepsilon$. Consider $\mathbf{S}_3^k$ again. $\mathbf{S}_3^k$ is a uniform bicubic B-spline surface patch with control mesh $\Pi_3^k$. Therefore, if $l_k$ recursive subdivisions are performed on the control mesh $\Pi_3^k$, by Lemma 2 and Lemma 3, we would have

$$\left\| \mathbf{L}_3^{l_k}(u,v) - \mathbf{S}_3^k(u,v) \right\| \leq \frac{1}{3} \left(\frac{1}{4}\right)^{l_k} Z_3^k, \tag{29}$$

where $\mathbf{L}_3^{l_k}(u,v)$ is a level-$l_k$ control mesh relative to $\Pi_3^k$ and $Z_3^k$ is the second order norm of $\mathbf{S}_3^k(u,v)$. Therefore, by combining the above result with Eqn. (26), we have

$$\left\| \mathbf{L}_3^{l_k}(u,v) - \mathbf{S}_3^k(u,v) \right\| \leq \frac{1}{3} \left(\frac{1}{4}\right)^{l_k} (W)^k Z_0^0, \tag{30}$$

We get the following Lemma by setting the right hand side of Eqn. (30) smaller than or equal to $\varepsilon$.

**Lemma 8:** In Lemma 7, if the distance between $\mathbf{S}_3^k$ and $\mathbf{L}_3^k$ is not smaller than $\varepsilon$, then one needs to perform $l_k$

$$l_k = \left\lceil \log_4\left(\frac{(W)^k Z_0^0}{3\varepsilon}\right) \right\rceil, \tag{31}$$

more recursive subdivisions on the level-$k$ control mesh $\Pi_3^k$ of $\mathbf{S}_3^k$ to make the distance between $\mathbf{S}_3^k$ and the level-$(k+l_k)$ control mesh smaller than $\varepsilon$.

This result works for $\mathbf{S}_1{}^k$ and $\mathbf{S}_2{}^k$ as well. Note that the value of $(W)^k Z_0^0$ is already computed in Lemma 7 and $W$ has to be computed only once. Therefore, the subdivision depth $D_k$ for $\mathbf{S}_1{}^k$, $\mathbf{S}_2{}^k$ and $\mathbf{S}_3{}^k$ is defined as follows:

$$D_k = \max\left\{ n_\varepsilon, k + \left\lceil \log_4\left( \frac{(W)^k Z_0^0}{3\varepsilon} \right) \right\rceil \right\} \tag{32}$$

Consequently, we have the following main theorem:

**Theorem 9:** Let $\prod_0^0 = \{ \mathbf{V}_i \mid 1 \le i \le 2N+8 \}$ be the control mesh of a CCSS patch $\mathbf{S}(u,v)$. The control points are ordered following Stam's fashion [12] with $\mathbf{V}_1$ being an extraordinary vertex of valence $N$ (see Fig. 1). For a given $\varepsilon > 0$, if we compute $n_\varepsilon$ as in Eqn. (23) and $D$ as follows:

$$D = \max \{ D_k \mid 1 \le k \le n_\varepsilon \}, \tag{33}$$

where $D_k$ is defined in Eqn. (32) then after $D$ recursive subdivisions, the distance between $\mathbf{S}(u,v)$ and the level-$D$ control mesh is smaller than $\varepsilon$.

## 4. EXAMPLES

Some examples of the presented distance evaluating and subdivision depth computing techniques are shown in this section. In Figs. 3(a), 3(b) and 3(c), the distances between the blue faces of the control meshes and the corresponding limit surface patches are 0.034, 0.15 and 0.25, respectively. For an error tolerance of 0.01, the subdivision depths computed for these mesh faces are 1, 22 and 24, respectively. The reason that the last two cases have large subdivision depths is because each of them has an extraordinary vertex. For the blue mesh face shown in Fig. 3(c), subdivision depths for error tolerances 0.25, 0.2, 0.1, 0.01, 0.001, and 0.0001 are 1, 3, 9, 24, 40, and 56, respectively.
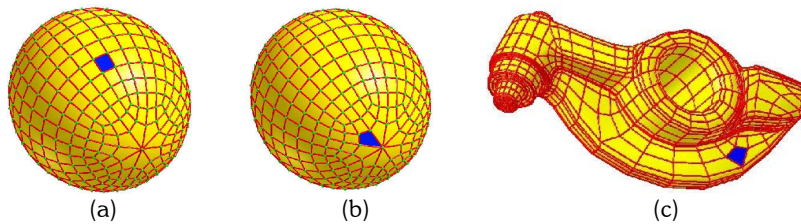


| (a) | (b) | (c) |

Fig. 3. Distance and subdivision depth computation for a CCSS patch with: (a) no extraordinary vertex, (b) an extraordinary vertex of valence 8, (c) an extraordinary vertex of valence 5.

## 5. CONCLUSIONS

A subdivision depth computation technique for CCSS's is presented. This technique provides a precision/error control tool for all tessellation based applications of subdivision surfaces.

One possible disadvantage of the subdivision depth computation technique is that it might generate a relatively large subdivision depth for a vicinity of an extraordinary vertex which is actually quite flat. This is because the first order norm can detect the location difference of two points, but not the difference between their curvatures. Therefore, even though two points are on the same plane, as far as they are far apart, a large $n_\varepsilon$ would still be generated by the subdivision depth computation process (see Theorem 6). A possible solution to this problem is to consider second order norm for $\mathbf{\Phi}_0{}^n$, $\mathbf{\Phi}_1{}^n$, $\mathbf{\Phi}_2{}^n$ and $\mathbf{\Phi}_3{}^n$ as well as the first order norm when computing $n_\varepsilon$ for the vicinity of an extraordinary vertex.

## 6. ACKNOWLEDGEMENTS

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