Best Multi-degree Reduction of Bernstein Polynomial in L_2-norm Based on an Explicit Termination Criterion

Renjiang Zhang[1,3], Guojin Wang[2] and Weiyin Ma[1]

[1]City University of Hong Kong, mewma@cityu.edu.hk
[2]Zhejiang University, gjwang@hzcnc.com
[3]China Jiliang University, zrj@cjlu.edu.cn

ABSTRACT

Based on the properties of orthogonal polynomials, we derive an explicit constrained degree reduction criterion for Bernstein-Bézier polynomials in L_2-norm. The criterion can be used to determine whether a further degree reduction can be applied to the polynomial in advance with a given tolerance ε. An efficient algorithm is also presented for obtaining the best Bernstein-Bézier polynomial after degree reduction. With the proposed algorithm, one can avoid the blind procedure for degree reduction and terminate the procedure in advance when the estimated error is larger than the given tolerance.

Keywords: Bernstein-Bézier polynomial, Jacobi polynomial, degree reduction, error estimate.

1. INTRODUCTION

Given a Bernstein-Bézier polynomial of degree *n* as

$$P_n(t) = \sum_{k=0}^{n} b_k B_k^n(t), \quad t \in [0,1], \tag{1.1}$$

where b_k 's are the Bézier ordinates and $B_k^n(t)$'s are Bernstein basis functions, the constrained multi-degree reduction of the given polynomial $P_n(t)$ is defined as the solution in finding a polynomial $\tilde{P}_m(t)$ of reduced degree $m < n$, such that they have equal derivatives up to $(r-1)$ -th and $(s-1)$ -th orders respectively at the endpoints, namely

$$P_n^{(k)}(0) = \tilde{P}_m^{(k)}(0), P_n^{(l)}(1) = \tilde{P}_m^{(l)}(1); \quad k = 0,1,\cdots,r-1; l = 0,1,\cdots,s-1. \tag{1.2}$$

The distance between $P_n(t)$ and $\tilde{P}_m(t)$ in L_p -norm is defined as

$$\left\| P_n(t) - \tilde{P}_m(t) \right\|_{L_p} = \left(\int_0^1 \left\| P_n(t) - \tilde{P}_m(t) \right\|^p dt \right)^{\frac{1}{p}}. \tag{1.3}$$

If the distance between $P_n(t)$ and $\tilde{P}_m(t)$ in a certain norm is minimal, then $\tilde{P}_m(t)$ is called the best polynomial after constrained $(n-m)$ multi-degree reduction in that norm.

There have been many publications [1],[3-15] focusing on the degree reduction of a Bézier curve in various norm, but most of them can only obtain approximate solutions for constrained $(n-m)$ multi-degree reduction for polynomials. Recently, further progress has been made in [1],[10-11],[13-15] for constrained multi-degree reduction of polynomials in the L_2 norm. Let $\tilde{P}_n(t)$ be the Bernstein-Bézier polynomial of degree n inversely obtained from $\tilde{P}_m(t)$ with degree elevation. Lutterkort et al [10-11] discovered that the problem of the best $(n-m)$ degree reduction of Bernstein-Bézier polynomials in L_2 -norm is equivalent to the problem of finding the best Euclidean approximation from the coefficient vector of $\tilde{P}_n(t)$ to the coefficient vector of the original Bernstein-Bézier polynomial $P_n(t)$. Based on the finding, they developed an algorithm for the best degree reduction of Bernstein-Bézier polynomials in L_2 -norm. However, end points constraints were not considered. Ahn et al [1],[14] extended the result for tackling the case with end points constraints. Zheng and Wang [15] used the method of perturbing Bézier coefficients to solve the best constrained degree reduction polynomials in the L_2 -norm. Zhang and Wang [13] further developed a method which can provide an explicit matrix expression of the best constrained degree reduction polynomial and a precise error formula.

In literature, one may also find algorithms for degree reduction based on other norms. When the L_∞ -norm [6],[9],[12] is used, thanks to the good approximation property of Chebyshev polynomials, the best single degree reduction of Bernstein polynomial can be easily obtained. However, there has been no reported work for other cases such as the best constrained degree reduction of polynomials. For degree reduction in the L_1 -norm, there are also some results [3],[5] used for degree reduction of interval Bézier curves. One may find some further discussions regarding degree reduction based on other norms in [4],[7],[8].

However, the above algorithms are still not perfect and there is still space for further improvement. For example, one can not precisely give the approximation error formula to determine whether a given polynomial can be degree reduced in advance with a given tolerance ε . In other words, the approximation error can only be estimated after the corresponding polynomial is obtained with the constrained degree reduction procedure. An interesting and practical question is then whether we can determine the existence of a polynomial with reduced degree satisfying the given tolerance using a simple algorithm before we actually do the conversion?

In this paper, we develop a new algorithm for optimal degree reduction of Bernstein-Bézier polynomial in L_2 -norm. It is shown that the above problem is satisfactorily solved. In the following we first present the algorithm based on the recursive integral formulae, and then derive the explicit, simple degree reduction criterion. Some examples will be introduced afterwards with further discussions.

2. THE TERMINATION CRITETION AND THE RELATED RECURSIVE ALGORITHM

Let us recall that a Jacobi polynomial $J_n^{(s,r)}(x)$ of degree n is an orthogonal polynomial on the weight function

$$\omega(x) = (1-x)^s(1+x)^r, x \in [-1,1], (s > -1, r > -1)$$

on $[-1,1]$ (see [2]), that is

$$\int_{-1}^{1}(1-x)^s(1+x)^r J_n^{(s,r)}(x)J_m^{(s,r)}(x)dx = \begin{cases} 0, & n \neq m; \\ \delta_n^{(s,r)}, & n = m. \end{cases} \tag{2.1}$$

where s, r are confined on non-negative integers for our application and

$$\delta_n^{(s,r)} = \frac{2^{r+s+1}}{2n+r+s+1} \cdot \frac{(n+r)!(n+s)!}{n!(n+r+s)!}. \tag{2.2}$$

A Jacobi polynomial of degree n , $J_n^{(s,r)}(x)(-1 \le x \le 1)$, can be expressed as (see [2])

$$J_n^{(s,r)}(x) = \sum_{k=0}^{n} \binom{n+s}{n-k} \binom{n+s+r+k}{k} \left(\frac{x-1}{2}\right)^k. \tag{2.3}$$

Note that $J_n^{(0,0)}(x)$ is a Legendre polynomial. Applying a transformation with $x = 2t-1$ and writing $\tilde{J}_i^{(s,r)}(t) = J_i^{(s,r)}(2t-1)$, one can obtain the constrained best polynomial $\tilde{P}_m(t)$ after ($n-m$)-degree reduction in the L_2 norm as follows [13]:

$$\tilde{P}_m(t) = P_n(t) - (1-t)^s t^r [\tilde{b}_{n-r-s}\tilde{J}_{n-r-s}^{(2s,2r)}(t) + \tilde{b}_{n-r-s-1}\tilde{J}_{n-r-s-1}^{(2s,2r)}(t) + \cdots + \tilde{b}_{m-r-s+1}\tilde{J}_{m-r-s+1}^{(2s,2r)}(t)]. \tag{2.4}$$

The exact error for this approximation is

$$E_{(n,n-m)}^{(s,r)} = \left\| P_n(t) - \tilde{P}_m(t) \right\|_{L_2} = \sqrt{\frac{\tilde{b}_{n-r-s}^2 \delta_{n-r-s}^{(2s,2r)} + \cdots + \tilde{b}_{m-r-s+1}^2 \delta_{m-r-s+1}^{(2s,2r)}}{2^{2s+2r+1}}}, \tag{2.5}$$

where $\tilde{b}_k$ for $k = m-r-s+1, \ldots, n-r-s$ are undetermined constants. In the following we determine $\tilde{b}_k$ by using a new recursive method.

Further recalling the orthogonal properties of Legendre polynomials, it is easy to see that

$$\int_{-1}^{1} q_i(t) J_i^{(0,0)}(t)dt = 0, \quad i = 0,1,2,\cdots,n-1 \tag{2.6}$$

where $q_i(t)$ is an arbitrary polynomial of degree i not large than n. Multiplying both sides of Eqn. (2.4) by $\tilde{J}_n^{(0,0)}(t)$, integrating from 0 to 1 and noting Eqn. (2.6), we have

$$\tilde{b}_{n-r-s} = \int_0^1 P_n(t)\tilde{J}_n^{(0,0)}(t)dt \Big/ \int_0^1 \tilde{\omega}(t) J_{n-r-s}^{(2s,2r)}(t)\tilde{J}_n^{(0,0)}(t)dt, \tag{2.7}$$

where $\tilde{\omega}(t)=(1-t)^s t^r$. Multiplying both sides of Eqn. (2.4) by $\tilde{J}_{n-1}^{(0,0)}(t)$ and integrating from 0 to 1, we have

$$\begin{aligned}\tilde{b}_{n-r-s-1} &= \int_0^1 P_n(t)\tilde{J}_{n-1}^{(0,0)}(t)dt \Big/ \int_0^1 \tilde{\omega}(t) J_{n-r-s}^{(2s,2r)}(t)\tilde{J}_{n-1}^{(0,0)}(t)dt \\ &-\tilde{b}_{n-r-s}\cdot \int_{-1}^1 \tilde{\omega}(t)\tilde{J}_{n-r-s}^{(2s,2r)}(t)\tilde{J}_{n-1}^{(0,0)}(t)dt \Big/ \int_{-1}^1 \tilde{\omega}(t)\tilde{J}_{n-r-s-1}^{(2s,2r)}(t)\tilde{J}_{n-1}^{(0,0)}(t)dt .\end{aligned} \tag{2.8}$$

In general, we have

$$\begin{aligned}\tilde{b}_{n-r-s-l} &= \int_{-1}^1 P_n(t)\tilde{J}_{n-l}^{(0,0)}(t)dt \Big/ \int_{-1}^1 \tilde{\omega}(t)\tilde{J}_{n-r-s}^{(2s,2r)}(t)\tilde{J}_{n-l}^{(0,0)}(t)dt - \cdots \\ &-\tilde{b}_{n-r-s-l+1}\cdot \int_{-1}^1 \tilde{\omega}(t) J_{n-r-s}^{(2s,2r)}(t) J_{n-l}^{(0,0)}(t)dt \Big/ \int_{-1}^1 \tilde{\omega}(t) J_{n-r-s-1}^{(2s,2r)}(t) J_{n-l}^{(0,0)}(t)dt, \quad \text{for } l \le n-m-1\end{aligned} \tag{2.9}$$

With the above recursive formulae, we can determine $\tilde{b}_k \quad (k=m-r-s+1,\ldots,n-r-s)$ one by one.

To improve the computation efficiency, we may compute the following integrals used in the above formulae in advance, which are

$$\int_0^1 B_i^n(t)\tilde{J}_{n-j}^{(0,0)}(t)dt, \qquad i,j=0,1,2,\cdots,n,$$

and

$$\int_0^1 \tilde{\omega}(t) J_{n-i}^{(2s,2r)}(t)\tilde{J}_{n-j}^{(0,0)}(t)dt = \int_0^1 (1-t)^s t^r J_{n-i}^{(2s,2r)}(t)\tilde{J}_{n-j}^{(0,0)}(t)dt, \quad i,j=0,1,2,\cdots,n.$$

It is easy to see that all the above integrals can be converted to the following integral form

$$\int_0^1 (1-t)^m t^n dt, \ \ m,n=0,1,2,\cdots,$$

which can be explicitly evaluated as

$$\int_0^1 (1-t)^m t^n dt = \frac{1}{m+n+1}\cdot\frac{1}{\binom{m+n}{m}}.$$

This leads to the explicit evaluation of the above two integrals as

$$\begin{aligned}\int_0^1 B_i^n(t)\tilde{J}_{n-j}^{(0,0)}(t)dt &= \int_0^1 \binom{n}{i}(1-t)^{n-i}t^i \sum_{k=0}^{n-j}\binom{n-j}{k}\binom{n-j+k}{k}(t-1)^k dt \\ =\binom{n}{i}\sum_{k=0}^{n-j}(-1)^k\binom{n-j}{k}\binom{n-j+k}{k}\int_0^1 (1-t)^{n+k-i}t^i dt &= \binom{n}{i}\sum_{k=0}^{n-j}(-1)^k\binom{n-j}{k}\binom{n-j+k}{k}\frac{1}{n+k+1}\cdot\frac{1}{\binom{n+k}{i}}\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}\int_0^1 \tilde{\omega}(t)\tilde{J}_{n-i}^{(2s,2r)}(t)\tilde{J}_{n-j}^{(0,0)}(t)dt &= \int_0^1 (1-t)^s t^r \cdot \sum_{k=0}^{n-i}\binom{n-i+2s}{n-i-k}\binom{n-i+2s+2r+k}{k}(t-1)^k \cdot \sum_{l=0}^{n-j}\binom{n-j}{l}\binom{n-j+l}{l}(t-1)^l dt \\ &= \sum_{k=0}^{n-i}\sum_{l=0}^{n-j}(-1)^{k+l}\frac{1}{s+k+l+r+1}\binom{n-i+2s}{n-i-k}\binom{n-i+2s+2r+k}{k}\binom{n-j}{l}\binom{n-j+l}{l}\binom{s+k+l+r}{r}.\end{aligned} \tag{2.11}$$

One can thus see that all the integrals in Eqn. (2.7)-(2.9) can be explicitly evaluated.

Based on the above formulae and the error expression (2.5), it is now easy to establish an explicit criterion for constrained Bernstein-Bézier polynomial degree reduction. Given an original Bernstein-Bézier polynomial (1.1), the orders r and s for the endpoint constraints (1.2), and a given tolerance ε, one can determine if there exists a reduced degree polynomial that can approximate the polynomial (1.1) satisfying the given constraints (1.2) in L_2-norm by firstly computing $\{\tilde{b}_{n-r-s-h}\}_{h=0}^{l}$ and then verifying $\{E_{(n,n-1-h)}^{(s,r)}\}_{h=0}^{l}$, $l=0,1,\cdots,n-m-1$. Furthermore, the number of degrees that can be reduced at most can also be deduced.

Let $m=n-1$ in Eqn. (2.4), we obtain the best constrained polynomial after one degree reduction

$$\tilde{P}_{n-1}(t)=P_n(t)-(1-t)^s t^r \tilde{b}_{n-r-s} J_{n-r-s}^{(2s,2r)}(2t-1), \tag{2.12}$$

where the constant $\tilde{b}_{n-r-s}$ is given explicitly as

$$\tilde{b}_{n-r-s}=(-1)^s \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k} b_k \bigg/ \binom{2n}{n-r-s}. \tag{2.13}$$

The above equation can be directly derived from Eqn. (2.7) or by comparing the coefficients of t^n of both sides of Eqn. (2.12). Following Eqn. (2.2) and Eqn. (2.5), we know that the exact error for the above one degree reduction approximation is

$$E_{(n,1)}^{(s,r)}=\left\|P_n(t)-\tilde{P}_{n-1}(t)\right\|_{L_2}=\sqrt{\frac{1}{2n+1}\cdot\frac{(n-s)!(n-r)!}{n!(n-r-s)!}}\cdot\left|\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}b_k\right| \bigg/ \binom{2n}{n-r-s}.$$

If the degree of a polynomial can be reduced, then it can be reduced by one at least. Thus, we obtain the following criterion which can be used to decide in advance whether or not the degree of a polynomial can be reduced:

Degree reduction criterion The degree of a Bernstein-Bézier polynomial (1.1) can be reduced under constraints (1.2) in L_2-norm with a given a tolerance ε if and only if

$$E_{(n,1)}^{(s,r)}=\sqrt{\frac{1}{2n+1}\cdot\frac{(n-s)!(n-r)!}{n!(n-r-s)!}}\cdot\left|\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}b_k\right| \bigg/ \binom{2n}{n-r-s}<\varepsilon. \tag{2.14}$$

One may notice that by letting $E_{(n,l)}^{(s,r)}=0$, it leads to

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}b_k=0$$

following equation (2.14). This is a condition under which a Bernstein-Bézier polynomial of degree n degenerates to a polynomial of degree $n-1$. Also following (2.14), it is easy to see that the error of degree reduction will be large for coefficients of Bernstein-Bézier polynomial with positive and negative alternate signs.

Based on the above discussions, we now give a degree reduction algorithm as follows:

Algorithm

Input: The degree of the original Bernstein-Bézier polynomial n, the ordinates of the Bernstein-Bézier polynomial $(b_0, b_1, \cdots, b_n)$, the end-point derivative constraints up to (s,r), and the required tolerance ε for degree reduction.

Output: A flag indicating whether the degree of the input polynomial can be reduced, and if yes, the best polynomial after degree reduction with the largest possible degrees.

Steps:

Step 1. Compute the left hand side of Eqn. (2.14).

Step 2. Evaluate whether or not the Eqn. (2.14) is valid, and if not, go to Step 6.

Step 3. For $l=2$ to $n-r-s$,

Apply recursive formulae (2.9) to compute $\tilde{b}_{n-r-s-l}$;

Apply Eqn. (2.5) to compute $E_{(n,n-l)}^{(s,r)}$;

Evaluate whether or not the inequality $E_{(n,n-l)}^{(s,r)}<\varepsilon$ is valid, and if not, go to Step 5.

Step 4. Output the best degree-reduced polynomial $\tilde{P}_{r+s-1}(t)$ (apply Eqn. (2.5)) and the precise error. Go to Step 7, i.e. the end of the algorithm.

Step 5. Compute and output the best degree-reduced polynomial $\tilde{P}_{n-r-s-l-1}(t)$ by applying Eqn. (2.5) and the precise error $E_{(n,n-l+1)}^{(s,r)}$. Go to Step 7, i.e. the end of the algorithm.

Step 6. Output a flag indicating that the degree of this polynomial can not be reduced.

Step 7. The end of the algorithm.

3. DEGREE REDUCTION OF BERNSTEIN BÉZIER CURVES

In this and the following sections, we present some examples on degree reduction of both Bernstein Bézier curves and parametric Bézier curves. For clarity, we introduce $e_{(n,n-m)}^{(r,s)}$ defined as

$$e_{(n,n-m)}^{(r,s)} = E_{(n,n-m)}^{(s,r)},$$

i.e., we commute the symbols of s and r such that (r,s) indicates the orders of the end constraints at parameters $t=0$ and $t=1$, respectively, on the illustrations.

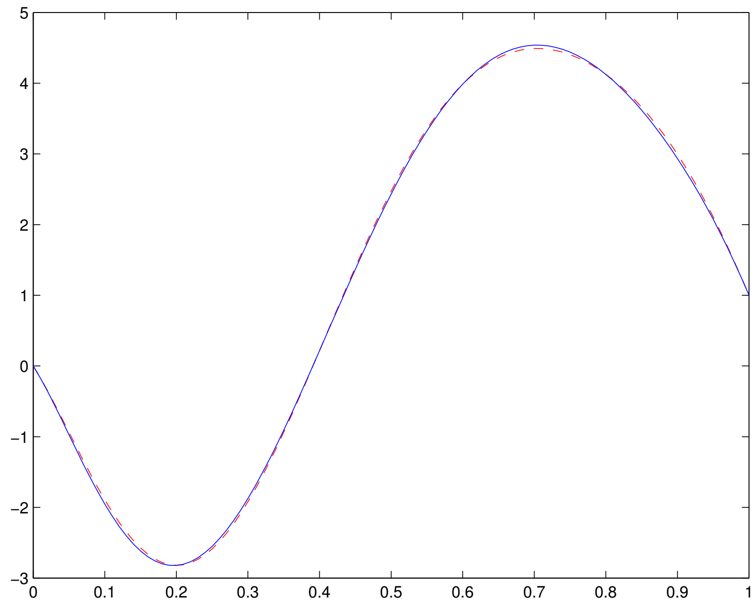

Fig. 1: A Bernstein polynomial of degree 8 (red broken line) and its best Bernstein polynomial after one degree reduction with endpoint interpolation constraints up to $(r,s)=(2,2)$, i.e. with the 1st order derivative match at both the left and the right sides.

Example 1: For a given tolerance $\varepsilon = 0.05$, determine whether or not the degree of the following Bernstein-Bézier polynomials can be reduced.

(1) the Bernstein-Bézier ordinates

$$(b_0,b_1,b_2,b_3,b_4,b_5,b_6) = (0,2,3,2,4,3,1) \quad \text{with} \quad (r,s) = (1,2)$$

(2) the Bernstein-Bézier ordinates

$$(b_0,b_1,b_2,b_3,b_4,b_5,b_6,b_7,b_8) = (0,-2,-7,-4,10,3,6,4,1) \quad \text{with} \quad (r,s) = (2,3)$$

(3) the Bernstein-Bézier ordinates

$$(b_0,b_1,b_2,b_3,b_4,b_5,b_6,b_7,b_8) = (0,-2,-7,-4,10,3,6,4,1) \quad \text{with} \quad (r,s) = (2,2)$$

Following the criterion, for a Bernstein-Bézier polynomial of (1.1) with degree $n=6$ and $(r,s)=(1,2)$, we have

$$e_{(6,1)}^{(1,2)}(0,2,3,2,4,3,1) = \sqrt{\frac{2}{39}\cdot\frac{9}{55}} \approx 0.037 < 0.05,$$

so the degree of the polynomial can be reduced for this tolerance ε.

For the Bernstein-Bézier polynomial of (1.2) with degree $n=8$ and $(r,s)=(2,3)$, we have

$$e_{(8,1)}^{(2,3)}(0,-2,-7,-4,10,3,6,4,1) = \sqrt{\frac{7}{2431}\cdot\frac{713}{560}} \approx 0.0683 > 0.05,$$

so the degree of the polynomial can not be reduced for the given tolerance ε.

For the Bernstein-Bézier polynomial of (1.2) with degree $n=8$ and $(r,s)=(2,2)$, we have

$$e_{(8,1)}^{(2,2)}(0,-2,-7,-4,10,3,6,4,1)=\sqrt{\frac{14}{1683}\cdot\frac{713}{1820}}\approx 0.0357<0.05,$$

so the degree of the polynomial can be reduced for the given tolerance ε. It is also easy to compute $\tilde{b}_4=\frac{713}{1820}$ from (2.13), and then get the best Bernstein polynomial after one degree reduction as

$$\tilde{P}_7(t)=P_8(t)-\frac{713}{1820}(1-t)^2t^2J_4^{(4,4)}(2t-1)=P_8(t)-\frac{713}{1820}(1-t)^2t^2\left[\sum_{k=0}^{4}\binom{8}{4-k}\binom{12+k}{k}(t-1)^k\right]$$

or in explicit Bernstein-Bézier form as

$$\tilde{P}_7(t)=-16(1-t)^6t-\frac{5393}{26}(1-t)^5t^2+\frac{20661}{130}(1-t)^4t^3+\frac{30441}{130}(1-t)^3t^4+\frac{2849}{26}(1-t)^2t^5+31(1-t)t^6+t^7.$$

The Bernstein polynomials before and after degree reduction are shown in Figure 1.

Example 2: For a given Bernstein polynomial of degree 7 with Bézier ordinates

$$(b_0,b_1,b_2,b_3,b_4,b_5,b_6,b_7)=(1,3,3,6,-2,6,4,2)$$

we want to find its best Bernstein polynomial after one degree reduction subjecting to endpoint interpolation constraints up to $(r,s)=(3,2)$ and a given tolerance $\varepsilon=0.01$. Following Eqn. (2.13), one can easily obtain $\tilde{b}_2=\frac{337}{91}$ and

$$e_{(7,1)}^{(3,2)}=\sqrt{\frac{1}{55}\cdot\frac{337}{273}}\approx 0.166>\varepsilon=0.01.$$

The above equation indicates that the degree of the polynomial can not be reduced with the given tolerance $\varepsilon=0.01$.

If the given tolerance is relaxed to $\varepsilon=0.17$, e.g., the degree of the polynomial can then be reduced at least by one as shown in the previous equation. One can further find that $\tilde{b}_1=-\frac{11}{17}$ and

$$e_{(7,2)}^{(3,2)}\approx 0.18>\varepsilon=0.17.$$

Thus the degree of the polynomial can only be reduced by one with $\varepsilon=0.17$ and the procedure is terminated. If we do not need to control the error when performing degree reduction, the degree of this polynomial can be further reduced down to degree 4 under the constraints $(r,s)=(3,2)$. The ordinates of the best polynomials after degree reduction to degrees 6, 5 and 4 are listed respectively below following the algorithm:

$$(b_0^1,b_1^1,b_2^1,b_3^1,b_4^1,b_5^1,b_6^1)=\left(1,\frac{10}{3},\frac{43}{15},\frac{5761}{1820},\frac{809}{273},\frac{13}{3},2\right)$$

$$(b_0^2,b_1^2,b_2^2,b_3^2,b_4^2,b_5^2)=\left(1,\frac{19}{5},\frac{12}{5},\frac{2576}{910},\frac{24}{5},2\right)$$

$$(b_0^3,b_1^3,b_2^3,b_3^3,b_4^3)=\left(1,\frac{9}{2},1,\frac{11}{2},2\right)$$

The Bernstein polynomials of degree 6, 5 and 4 before and after the degree reduction are shown respectively in Figure 2, 3 and 4.

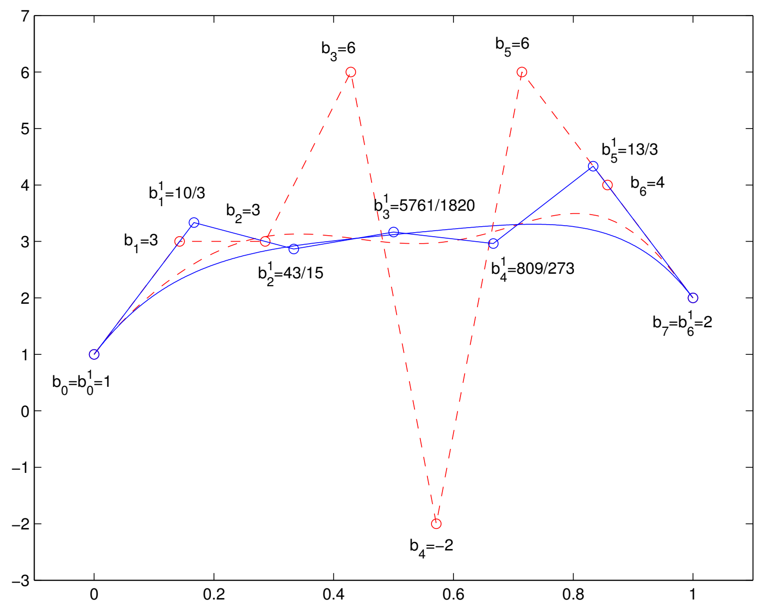

Fig. 2: A Bernstein polynomial of degree 7 (red broken line) and its best Bernstein polynomial after one degree reduction with endpoint interpolation constraints up to $(r,s)=(3,2)$, i.e. with the 2nd and the 1st order derivative match at the left and the right sides, respectively.

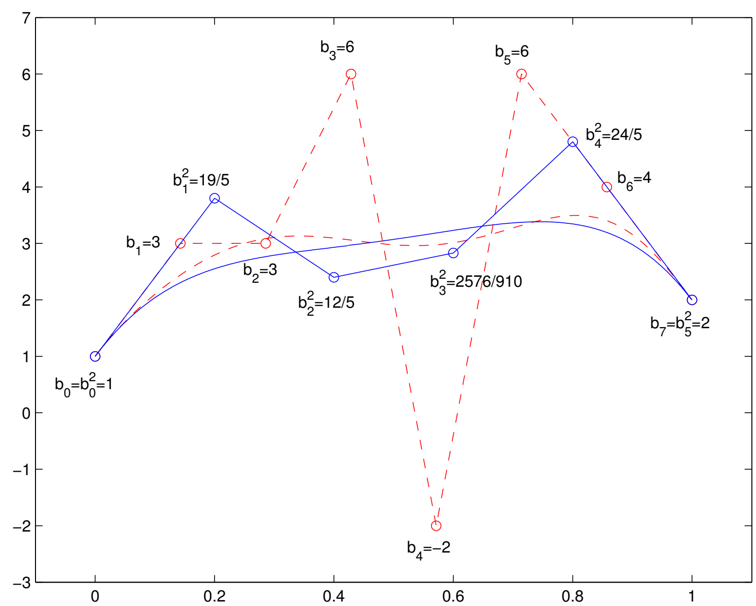

Fig. 3: A Bernstein polynomial of degree 7 (red broken line) and its best Bernstein polynomial after two degree reduction with endpoint interpolation constraints up to $(r,s)=(3,2)$, i.e. with the 2nd and the 1st order derivative match at the left and the right sides, respectively.

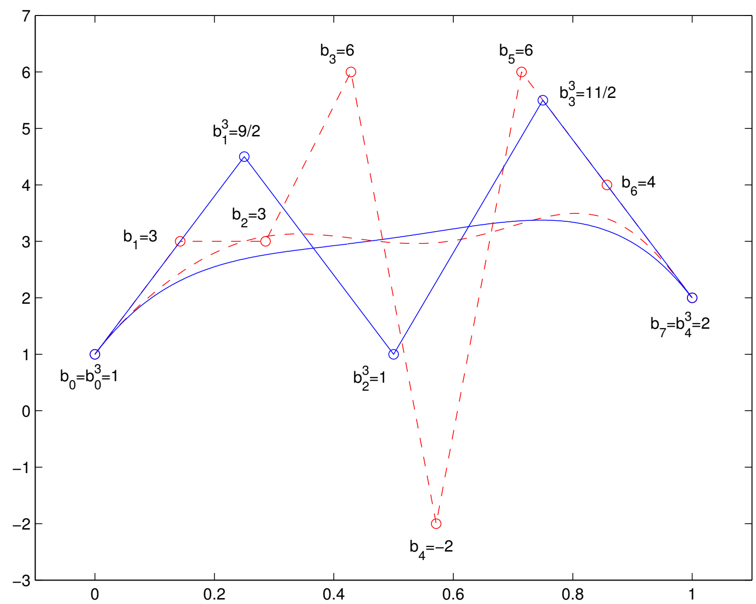

Fig. 4: A Bernstein polynomial of degree 7 (red broken line) and its best Bernstein polynomial after three degree reduction with endpoint interpolation constraints up to up to $(r,s)=(3,2)$, i.e. with the 2nd and the 1st order derivative match at the left and the right sides, respectively.

The above example shows that after obtaining the constant $\tilde{b}_{n-j}$ one may first compute the error before obtaining the next $\tilde{b}_{n-j-1}\left(j\in[r+s,r+s+n-m-1]\right)$. So one can terminate the degree reduction procedure if the error is large than the given tolerance. This can avoid the blind procedure for degree reduction.

4. DEGREE REDUCTION OF PARAMETRIC BÉZIER CURVES

A parametric Bézier curve of degree n can be expressed as

$$\mathbf{P}(t)=\sum_{k=0}^{n}\mathbf{b}_k B_k^n(t)\ ,\ \ t\in[0,1]$$

where $\mathbf{b}_k(x_{1,k},x_{2,k},\cdots,x_{i,k})\in R^i,(i=1,2,\cdots)$. We can apply the above degree reduction algorithm to the i-th coordinate of $\mathbf{P}(t)$:

$$P_i(t)=\sum_{k=0}^{n}x_{i,k}B_k^n(t)\ ,\ \ t\in[0,1]$$

Denoting the error of component $P_i(t)$ after degree reduction as e_i, we can then define the overall error of $\mathbf{P}(t)$ after degree reduction as $e=\sqrt{e_1^2+e_2^2+\cdots}$, , which can be used as the criterion for degree reduction of Bézier curves in i-dimensional space.

Example 3: For a given planar Bézier curve of degree 5 with Bézier control points as follows

$$\{\mathbf{b}_0,\mathbf{b}_1,\mathbf{b}_2,\mathbf{b}_3,\mathbf{b}_4,\mathbf{b}_5\}=\{(-1,0),(0,1),(2,4),(3,2),(5,5),(7,0)\}$$

and with endpoints interpolation constraints with orders $(r,s)=(1,2)$. Applying the algorithm to the individual coordinates of the control points, respectively, we obtain the updated Bézier curve with the following set of new control points after one degree reduction

$$\{\mathbf{b}_0^1,\mathbf{b}_1^1,\mathbf{b}_2^1,\mathbf{b}_3^1,\mathbf{b}_4^1\}=\left\{(-1,0),\left(\frac{29}{60},\frac{31}{12}\right),\left(\frac{43}{18},\frac{25}{18}\right),\left(\frac{9}{2},\frac{25}{4}\right),(7,0)\right\},$$

where, the superscript 1 indicates that they are control points obtained after one degree reduction. The Bézier curves before and after degree reduction are shown in Figure 5.

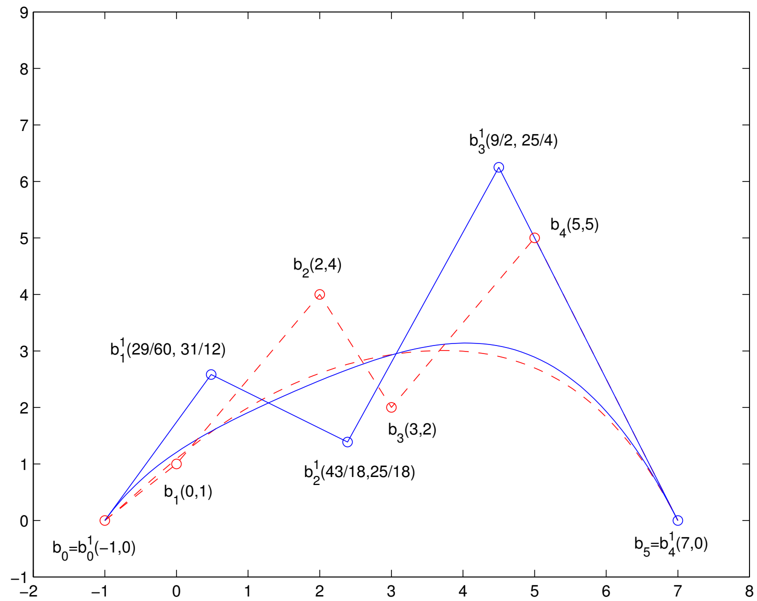

Fig. 5: A Bézier planar curve of degree 5 (red broken line) and the updated curve of degree 4 after one degree reduction with endpoint interpolation constraints up to $(r,s)=(1,2)$, i.e. with function and the 1st order derivative match at the left and the right sides, respectively.

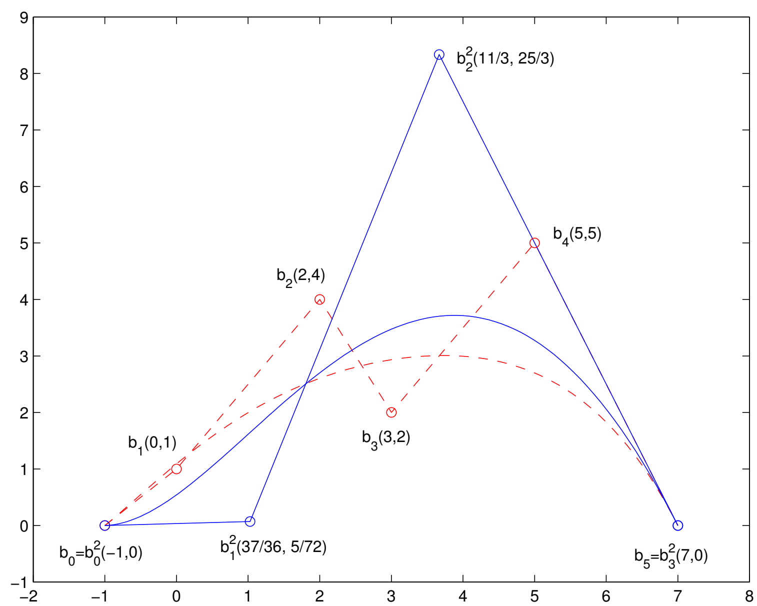

Fig. 6: A Bézier planar curve of degree 5 (red broken line) and the updated curve of degree 3 after two degree reduction with endpoint interpolation constraints up to $(r,s)=(1,2)$, i.e. with function and the 1st order derivative match at the left and the right sides, respectively.

Similarly, we can also obtain the updated Bézier curve with the following set of new control points after two degree reduction

$$\left\{\mathbf{b}_0^2, \mathbf{b}_1^2, \mathbf{b}_2^2, \mathbf{b}_3^2\right\} = \left\{(-1,0), \left(\frac{37}{36}, \frac{5}{72}\right), \left(\frac{11}{3}, \frac{25}{3}\right), (7,0)\right\},$$

The Bézier curves before and after degree reduction are shown in Figure 6.

5. CONCLUSIONS

This paper presents an error criterion in explicit form for constrained degree reduction of polynomials in L_2-norm. Using this criterion, it is possible to calculate the number of degrees for constrained multiple-degree reduction in advance with a given tolerance. The explicit error criterion is formulated in a recursive fashion and it avoids the blind procedure for degree reduction. Some examples are presented on both degree reduction of Bernstein Bézier curves and degree reduction of parametric Bézier curves.

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