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Interactive Control of Planar Class A Bézier Curves using Logarithmic Curvature Graphs

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ABSTRACT

We present a method for interactively drawing a planar class A Bézier curve segment. First, we present a method for interactively drawing a typical class A Bézier curve segment by specifying three points like a quadratic Bézier curve segment. We show that as the degree of a typical class A Bézier curve segment is elevated, the curve converges to a logarithmic spiral segment. At the limit of infinite degree, the curve segment becomes a logarithmic spiral segment. We also present a method for drawing a general class A Bézier curve segment by perturbing the elements of the typical class A matrix so that the endpoint constraints are satisfied. To see the characteristics of the generated curves, we propose to use logarithmic curvature graphs.

Keywords: class A Bézier curves, monotone curvature, logarithmic spiral, curvature graph.
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1. INTRODUCTION

For highly aesthetic shape design, such as the design of car bodies, it is very important to use aesthetic curves as key lines. Usually, the curvature distributions of such curves are of primal importance and the curvature of a curve segment is required to be monotonically varying[3]. Harada insists that a tighter restriction should be given for aesthetic curves based on his analysis of many aesthetic curves in artificial and the natural world[8,16]. The restriction is the linearity of logarithmic curvature graphs (formerly called logarithmic curvature histograms). Miura derived the general formula of log-aesthetic curves (formerly called Aesthetic Curves) with linear logarithmic curvature graphs[10]. Yoshida and Saito clarified the overall shapes of log-aesthetic curves and presented a method for interactively drawing a curve segment[14]. Yoshida and Saito also proposed Quasi-log-aesthetic curves (formerly called Quasi-Aesthetic Curves[15]) that approximately represent log-aesthetic curves in rational cubic Bézier forms. In practical situations, however, strict linearity of the logarithmic curvature graph may be too restrictive. Curves with looser restriction (approximate linearity) or the curves that can control the linearity of their logarithmic curvature graphs are desired.

In this paper, we investigate a class of curves with monotonically varying curvature, which are planar class A Bézier curves. Class A Bézier curves[4] are curves with monotonically varying curvature and torsion proposed by Farin. Though an interactive generation method for *typical* class A Bézier curves (see Section 3) is known, the interactive control method of general class A Bézier curves is not known. We first present a method for interactively controlling a typical class A Bézier curve segment by three points like a quadratic Bézier curve. We show that as the degree of typical class A Bézier curve is elevated, the curve converges to a logarithmic spiral. A general class A Bézier curve segment is interactively generated by perturbing the class A matrix so that endpoint constraints (the positions and the tangent vectors) are satisfied. To see the characteristics of the generated curves, we propose to use logarithmic curvature graphs.

The rest of this paper is organized as follows. Section 2 reviews relevant literature. Section 3 reviews class A Bézier curves. Section 4 describes logarithmic curvature graphs and their characteristics. In Section 5, we present a method for interactively controlling typical class A Bézier curves by specifying three points like a quadratic Bézier curves. Then in Section 6, we show that as the degree of a typical class A Bézier curve is elevated, the curve converges to a logarithmic spiral. Section 7 presents a method for interactively controlling general class A Bézier curves. Section 8 shows the generated class A Bézier curves and their logarithmic curvature graphs. Finally, conclusions are presented in Section 9.

2. RELATED WORKS

Free-form curves and surfaces, such as Bézier or NURBS, are widely used in current CAD systems. From the viewpoint of designing aesthetically appealing curves, such as the key lines of car bodies, such free-form curves are very difficult to manipulate. For aesthetically appealing curve segments, a designer may want to modify the curve shape under the constraint of monotonically varying curvature. However, the curve shapes of Bézier or NURBS are usually controlled by their control points (and weights in case of rational curves and knots in case of NURBS). It is not an easy task to place or move control points such that the curve segment has monotonically varying curvature.

Several papers have dealt with the generation of Bézier or B-spline curves with monotonically varying curvature. Sapidis and Frey presented the necessary and sufficient condition of monotone curvature for quadratic polynomial Bézier curves [12]. Frey and Field presented the condition for rational quadratic curves [6]. Diez and Piper used precomputed tables so that polynomial cubic Bézier curves have monotonically varying curvature [2]. Wang et al. presented sufficient monotone curvature conditions for polynomial Bézier and B-spline curves of degree n [13]. More recently, Farin has proposed class A Bézier curves [4]. Cao and Wang presented the correct condition for class A Bézier curves generated by symmetric matrices[1]. The method for interactively drawing class A Bézier curve segments is not known except for the curve is a typical curve. Class A Bézier curves are reviewed in the next section.

Yoshida and Saito proposed log-aesthetic curves [14], which are formerly called Aesthetic Curves. Log-aesthetic curves are curves with monotone curvature and are based on the linearity of the logarithmic curvature graphs. This is based on the observation by Harada [8,16] that many of aesthetic curves in artificial and the natural world are curves whose logarithmic curvature graphs can be approximated by straight lines. We call the slope of the straight line α. When $\alpha = -1,1,2,\pm\infty$, log-aesthetic curves become the Clothoid, the logarithmic spiral, the circle involute, and the circle, respectively. When $\alpha = 0$, we recently found that the log-aesthetic curve becomes the Nielsen's spiral [7]. Log-aesthetic curves can be considered as the generalization of these curves. A log-aesthetic curve segment can be interactively drawn by specifying three points like a quadratic Bézier curve when α is specified [14]. Quasi-log-aesthetic curve segments in rational cubic Bézier forms have also been proposed [15]. In a Quasi-log-aesthetic curve segment, the linearity of the straight line in the logarithmic curvature graph is approximately preserved and its monotonicity of the curvature is guaranteed. We use logarithmic curvature graphs for the analysis of generated curves.

3. CLASS A BÉZIER CURVES

In this section, we review class A Bézier curves [4] and point out that a typical class A Bézier curve of degree 3 is the special case of a Pythagorean hodograph curve [5]. Though class A Bézier curves include space curves with monotone curvature and torsion, we only deal with planar class A Bézier curves in this paper. Cao and Wang recently found that the monotonicity of the curvature of class A Bézier curves is not proved [1]. Though they provided a proof for symmetric matrices, more investigation is necessary for the proof of general matrices. However, the monotonicity of the curvature can be easily checked by sampling the point of a curve and computing the curvatures. The class A condition, though proved only for symmetric matrices, is still useful for generating the curves with monotone curvature. In our implementation, we generate curves using the class A condition for symmetric matrices and the monotonicity of the curvature is always checked by sufficiently sampling the points on the curve and computing the curvatures. For planar curves, we found that the class A condition for symmetric matrices fails rarely for general matrices. If a correct class A condition for general matrices is found, our framework for interactively controlling class A Bézier curves will still work just by replacing the class A condition by the new one.

A planar Bézier curve $\mathbf{x}(t)$ of degree n is defined by

$$\mathbf{x}(t)=\sum_{i=0}^{n}B_i^n(t)\mathbf{b}_i \qquad (3.1)$$

where $B_i^n(t)$ is a Bernstein polynomial and $\mathbf{b}_i$ are two-dimensional control point vectors. Let $\Delta\mathbf{b}_i=\mathbf{b}_{i+1}-\mathbf{b}_i(0\le i\le n-1)$ be the forward difference vector of the control points. Using a class A matrix M defined below, class A Bézier curves are defined by giving the following constraints for the control point vectors.

$$\Delta\mathbf{b}_i=M^i\Delta\mathbf{b}_0 \qquad (i=0\ldots,n-1) \qquad (3.2)$$

Class A matrix M is a matrix such that for any vector $|\mathbf{v}|=1$ and for any $t\in[0,1]$, the following relationship holds:

$$|(1-t)\mathbf{v}+\mathrm{t}M\mathbf{v}|\ge|\mathbf{v}|\,. \qquad (3.3)$$

This means that the line segment defined by $\mathbf{v}$ and $M\mathbf{v}$ for any $|\mathbf{v}|=1$ does not intersect the circle whose radius is 1 except for its endpoint. See Fig. 1.

For a matrix M to satisfy Eqn. (3.3) for any vector $|\mathbf{v}|=1$ and for any $t\in[0,1]$, the following two conditions must holds.

(1) The angle between $\mathbf{v}$ and $M\mathbf{v}$ must be less than 90 degrees.
(2) The matrix M must be an expansion matrix that transforms every point of the unit sphere to the outside of the sphere.

To satisfy condition (1), Farin showed that the following matrices must have nonnegative eigenvalues [4]:

$$M^{\mathrm{T}}+M-2I,\quad M^{\mathrm{T}}M-I \qquad (3.4)$$

where I is the identity matrix. To satisfy condition (2), the singular values $\sigma_1,\sigma_2(\sigma_1\ge\sigma_2)$ of M must be greater than or equal to 1. To prove the monotonicity of the curvature, Cao and Wang gives the following requirement for σ_1,σ_2 of symmetric matrices [1]:

$$2\sigma_1\ge\sigma_2+1,\quad 2\sigma_2\ge\sigma_1+1\,. \qquad (3.5)$$

Thus, for a symmetric matrix M to be class A, the matrices of Eqn.(3.4) must have nonnegative eigenvalues, the singular values σ_1,σ_2 of M must be greater than or equal to 1 and Eqn.(3.5) must hold. We call these conditions for matrix M *class A conditions*. Again, this class A condition is correct only for symmetric matrices M. We use the condition for general matrices and for guaranteeing the monotonicity of the curvature we sample the points on the curve and compute the curvatures. For planar curves, we found that the class A condition fails rarely for general matrices.

Farin gives an example of class A matrices, which we call *typical class A matrices*. *Typical class A Bézier curves* generated by typical class A matrices are the typical curves of Mineur[11], which was inspired by the curves of Higashi[9]. If the matrix M is a transformation matrix composed of a rotation by an angle $\theta<\pi/2$ followed by a scaling s, and the inequality

$$\cos\theta>\frac{1}{s}\ (\text{if } s>1)\quad \text{or}\quad \cos\theta>s\ (\text{if } s<1) \qquad (3.6)$$

holds, the matrix M becomes class A. The latter inequality ($\cos\theta>s$) can be derived by replacing $\mathbf{b}_i$ with $\mathbf{b}_{n-i}$ for $i=0,\ldots,n$. In contrast, a *general class A matrix* is a matrix that satisfies condition (1) and (2). General class A matrices

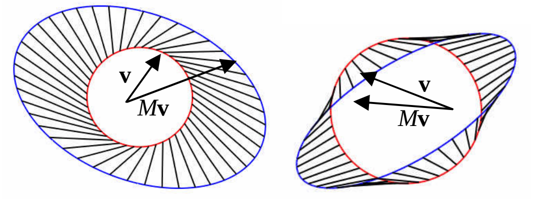
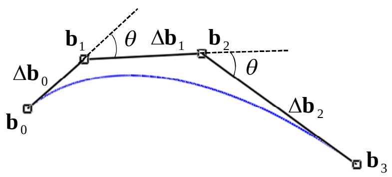


Fig. 1: An example of the action of Class A matrix(left) and NOT class A matrix (right).



$$\frac{\Delta\mathbf{b}_1}{\Delta\mathbf{b}_0}=\frac{\Delta\mathbf{b}_2}{\Delta\mathbf{b}_1}=s$$
$$\theta<\frac{\pi}{2}$$
$$\cos\theta>\frac{1}{s}$$

Fig. 2: A Typical Class Bézier Curve.

include typical class A matrices, and we call the generated curve the *general class A curve*. See Fig. 2 for an example of typical class A Bézier curves. Note that a typical class A Bézier curve of degree 3 is the special case of Pythagorean hodograph curves [5] of degree 3. If a Pythagorean hodograph curve of degree 3 satisfies Eqn. (3.6), the curve becomes a class A Bézier curve.

For a typical class A Bézier curves, Farin gives a method for interactive control by specifying two endpoints and their tangents. However, the method for interactively generating general class A Bézier curves is not known.

4. LOGARITHMIC CURVATURE GRAPHS

Log-aesthetic curves are curves whose logarithmic curvature graphs are represented by straight lines. Logarithmic curvature graphs were originally called logarithmic curvature histograms in [10,14,15], which are Miura's interpretation[10] of original logarithmic curvature histograms of Harada. Note that Harada's original logarithmic curvature histograms[8,16] were actually histograms.

Let ρ be the radius of curvature and s be the arc length. We assume that ρ is monotonically increasing with respect to s. The fundamental equation of log-aesthetic curves [10,14] is

$$\log\left(\rho \frac{\mathrm{d}s}{\mathrm{d}\rho}\right) = \alpha \log \rho + c \tag{4.1}$$

where α is the slope of the straight line in the logarithmic curvature graph and c is a constant. Since s is a monotonically increasing function of ρ, $\mathrm{d}s/\mathrm{d}\rho$ of log-aesthetic curves are always positive. However, $\mathrm{d}s/\mathrm{d}\rho$ of other curves may become negative. Thus we use $\log(\rho\,|\mathrm{d}s/\mathrm{d}\rho|)$ for the vertical axis of the logarithmic curvature graph. Taking the absolute value of $\mathrm{d}s/\mathrm{d}\rho$ means the reparameterization of the curve with respect to s. For the horizontal axis, $\log\rho$ is simply used. From Eqn.(4.1), we can derive the function of the curvature with respect to the arc length. See [14] for the details of the derivation. Eqn.(4.2) fixes the curvature function:

$$\kappa = \begin{cases} e^{-\Lambda s} & \text{if } \alpha = 0 \\ (\Lambda \alpha s + 1)^{-\frac{1}{\alpha}} & \text{otherwise} \end{cases} \tag{4.2}$$

where κ is the curvature and Λ is $\mathrm{d}\rho/\mathrm{d}s$ at $\rho = 1$. Therefore, the linearity of the logarithmic curvature graph means that the curvature of the curve is represented by a simple function of the arc length.

We can compute the logarithmic curvature graphs for arbitrary parametric curves, such as Bézier or NURBS curves. The horizontal value $\log\rho$ can be easily computed. The vertical value $\log(\rho\,|\mathrm{d}s/\mathrm{d}\rho|)$ can be computed using

$$\frac{\mathrm{d}\rho}{\mathrm{d}s} = -\frac{1}{\kappa^2}\frac{\mathrm{d}\kappa}{\mathrm{d}s} \tag{4.3}$$

and

$$\frac{d\kappa}{ds} = \frac{\det\left(\dot{\mathbf{x}}, \mathbf{x}^{(3)}\right)\dot{\mathbf{x}} \cdot \dot{\mathbf{x}} - 3\det\left(\dot{\mathbf{x}}, \ddot{\mathbf{x}}\right)\dot{\mathbf{x}} \cdot \ddot{\mathbf{x}}}{|\dot{\mathbf{x}}|^6}. \tag{4.4}$$

Fig. 3 shows three cubic Bézier curve segments with their curvature plots and their logarithmic curvature graphs. In the figure (and other figures throughout the paper), a white circle or a black circle is marked to denote the start point or the end point of a segment, respectively. Fig. 3(a) shows a curve segment with not monotonically varying curvature. Fig. 3(b) shows a curve segment with monotonically varying curvature but not linear logarithmic curvature graph. Fig. 3(c) shows a curve segment with monotonically varying curvature and linear logarithmic curvature graph.

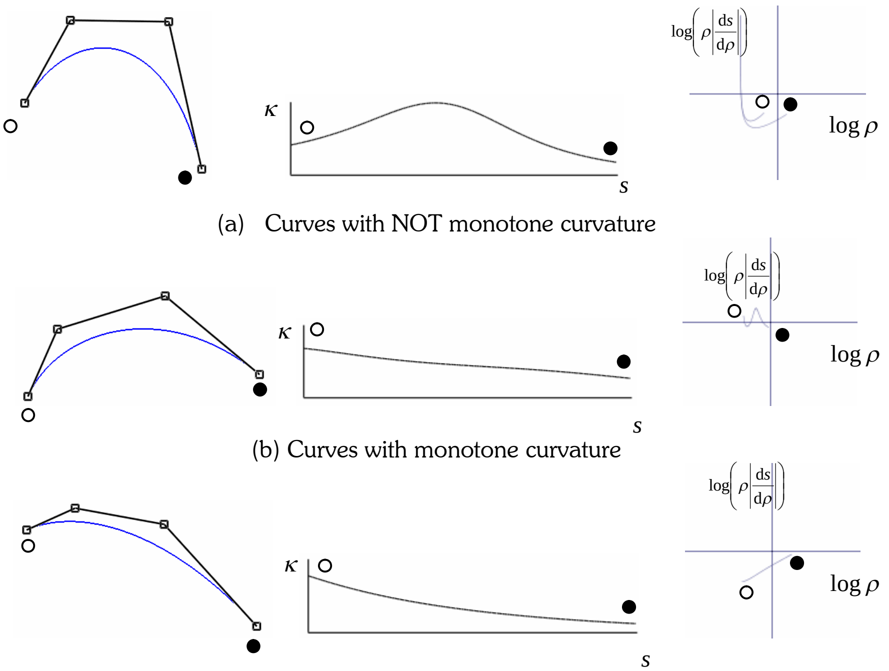
Logarithmic curvature graphs can be used to see the characteristics of the curves. Logarithmic curvature graphs have the following characteristics.

(a) Seeing the linearity of the logarithmic curvature graph gives some clue how the curve is deviated from log-aesthetic curves. In other words, if the logarithmic curvature graph is almost linear, the curvature can be approximately represented by a simple function of the arc length.
(b) If the horizontal value of the logarithmic curvature graph is monotonically increasing or decreasing, the curvature of the curve is monotonically varying.

(c) If the curvature of a curve is not monotonically varying or constant, the curve includes a point at $\mathrm{d}\kappa/\mathrm{d}s = 0$. At the point, $\log(\rho\,|\mathrm{d}s/\mathrm{d}\rho|)$ becomes infinite.

(d) $\log\rho$ of the inflection point of a curve is infinite. $\log(\rho\,|\mathrm{d}s/\mathrm{d}\rho|)$ of the inflection point is either infinite or undefined.

From the characteristics (c) and (d), the logarithmic curvature graph is not good at curves with not monotonically varying curvature or constant curvature, or curves with an inflection point. However, we can check the proximity of the curves to log-aesthetic curves. The monotonicity of the curvature can also be checked.



(a) Curves with NOT monotone curvature

(b) Curves with monotone curvature

(c) Curves with monotone curvature and approximately linear logarithmic curvature graph

Fig. 3: Cubic Bézier curves (left), their curvature plot (middle) and their logarithmic curvature graphs (right).

5. INTERACTIVE CONTROL OF TYPICAL CLASS A BÉZIER CURVES

For interactively controlling typical class A Bézier curves, we present a method for drawing typical class A Bézier curves of degree n by specifying three control points like a quadratic Bézier curve. Typical class A matrix M is

$$M = \begin{bmatrix} s\cos\theta & -s\sin\theta \\ s\sin\theta & s\cos\theta \end{bmatrix} \tag{5.1}$$

that satisfies Eqn.(3.6).

Let $\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2$ be three point vectors for generating a typical class A Bézier curve segment of degree $n(\geq 3)$. We would like to generate a curve whose endpoints are $\mathbf{a}_0$ and $\mathbf{a}_2$ and the tangent vectors at the endpoints are parallel to $\mathbf{a}_1 - \mathbf{a}_0$ and $\mathbf{a}_2 - \mathbf{a}_1$, respectively. Since $\mathbf{b}_0 = \mathbf{a}_0$ and $\mathbf{b}_n = \mathbf{a}_2$, we need to find $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{n-1}$. θ in Eqn.(5.1) is the angle between the vector $\mathbf{a}_1 - \mathbf{a}_0$ and the vector $\mathbf{a}_2 - \mathbf{a}_1$ divided by $n-1$. See Fig. 4. Let $b_0 = |\Delta\mathbf{b}_0|$ and $\mathbf{u} = \dfrac{\mathbf{a}_1 - \mathbf{a}_0}{|\mathbf{a}_1 - \mathbf{a}_0|}$.

We need to find b_0 and s such that

$$\sum_{j=0}^{n-1} b_0 M^j \mathbf{u} = (\mathbf{a}_2 - \mathbf{a}_0) \tag{5.2}$$

We use the optimization process to find b_0 and s such that

$$\mathbf{f}(b_0, s) = \sum_{j=0}^{n-1} b_0 M^j \mathbf{u} - (\mathbf{a}_2 - \mathbf{a}_0) \tag{5.3}$$

becomes the zero vector. Since we can compute the derivatives of Eqn.(5.3) with respect to b_0 and s, we can use the optimization using derivatives. Note that when $n = 3$, Eqn.(5.3) becomes of degree 2. Thus, b_0 and s can be uniquely computed without using the optimization. Once b_0 and s are computed, all the other control point vectors are computed by

$$\mathbf{b}_i = \mathbf{b}_0 + \sum_{j=0}^{i-1} b_0 M^j \mathbf{u} \quad (0 < \mathrm{i} \le \mathrm{m}). \tag{5.4}$$

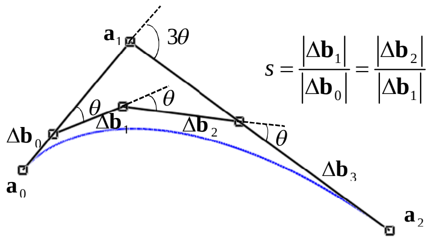


Fig. 4: Interactive control of a typical class A Bézier curve.

6. RELATIONSHIP WITH LOGARITHMIC SPIRALS

In this section, we show that typical class A Bézier curves are approximation to logarithmic spirals. More specifically, we show that as the degree of a typical class A Bézier curve is elevated, the curve converges to a logarithmic spiral.

A logarithmic spiral is a kind of spiral curves which appears in nature, such as a nautilus shell and spiral galaxies. Logarithmic spirals are included in log-aesthetic curves as the case of $\alpha = 1$. The equation of a logarithmic spiral whose pole $\mathbf{p}$ is at the origin on the complex plane is

$$\mathbf{LS}(\theta) = r_0 e^{k\theta i} \tag{6.1}$$

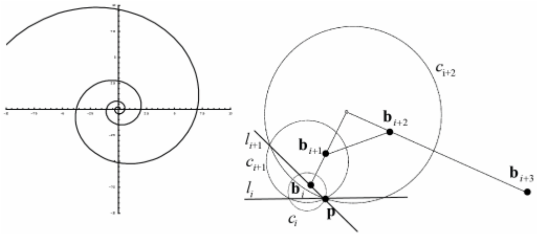
where i is the imaginary unit, $r_0 (> 0)$ and $k (0 \le \cot(k) \le \pi)$ are constants. The spiral curve rotates around the pole $\mathbf{p}$ of the spiral. Fig. 5(a) shows an example of a logarithmic spiral. The tangential angle θ is the angle between x-axis and the vector starting from the pole to $\mathbf{LS}(\theta)$. The distance from the pole to the point of the curve $\mathbf{LS}(\theta)$ is $|\mathbf{LS}(\theta)| = r_0 e^{k\theta}$.

We consider two points on a logarithmic spiral whose tangential angles are θ_0 and $\theta_0 + \theta_d$ ($\theta_d > 0$), respectively. The ratio of $|\mathbf{LS}(\theta_0 + \theta_d)|$ and $|\mathbf{LS}(\theta_0)|$ is

$$\frac{|\mathbf{LS}(\theta_0 + \theta_d)|}{|\mathbf{LS}(\theta_0)|} = e^{k\theta_d} = \text{const.} \tag{6.2}$$

Therefore, if the tangential angle is increased by θ_d, the distance between the pole to the curve is increased by a factor of $e^{k\theta_d}$. Conversely, for a certain curve, if we could show that the distance between the pole and the point of the curve is increased by a factor of $e^{k\theta_d}$ when the tangential angle is increased by θ_d, we can say that the curve is a logarithmic spiral since we can easily find all the parameters of Eqn.(6.1). We use this property to show that typical class A Bézier curves are approximation to logarithmic spirals.

We are given the control points $\mathbf{b}_i (0 \le i \le n)$ of a class A Bézier curve of degree n. We want to show that we can uniquely determine a logarithmic spiral that goes through all the control points. For control points $\mathbf{b}_i, \mathbf{b}_{i+1}, \mathbf{b}_{i+2}$ of a class A Bézier curve, we would like to find the pole $\mathbf{p}$ of a logarithmic spiral such that the triangles $\mathbf{p}\mathbf{b}_i\mathbf{b}_{i+1}$ and $\mathbf{p}\mathbf{b}_{i+1}\mathbf{b}_{i+2}$ are similar. If we could find such a pole for arbitrary i, then $|\mathbf{b}_{i+1} - \mathbf{p}| / |\mathbf{b}_i - \mathbf{p}| = |\mathbf{b}_{i+2} - \mathbf{b}_{i+1}| / |\mathbf{b}_{i+1} - \mathbf{b}_i| = s$ holds and there exists a logarithmic spiral that goes through all the control points. Here s is the scaling factor of the typical class A matrix. The pole exists on the Apollonius circle, which is the locus of point c whose distance from $\mathbf{b}_{i+1}$ is a multiple s of its distance from $\mathbf{b}_i$.



(a) a logarithmic spiral (b) Apollonius circles

Fig. 5: A logarithmic spiral and Bezier control points with Apollonius circles.

To compute the pole $\mathbf{p}$, we first compute an Apollonius circle c_i from $\mathbf{b}_i$ and $\mathbf{b}_{i+1}$. The Apollonius circle c_i is the locus of points $\mathbf{q}$ such that $|\mathbf{b}_{i+1}-\mathbf{q}|/|\mathbf{b}_i-\mathbf{q}|=s$, where $s=|\mathbf{b}_{i+2}-\mathbf{b}_{i+1}|/|\mathbf{b}_{i+1}-\mathbf{b}_i|$. In a similar manner, we can construct Apollonius circles c_{i+1}, c_{i+2} from $\mathbf{b}_{i+1}, \mathbf{b}_{i+2}$ and $\mathbf{b}_{i+2}, \mathbf{b}_{i+3}$, respectively. From the circles c_i and c_{i+1}, we can construct a line l_i that goes through the intersection points of the two circles. Similarly, we can construct another line l_{i+1} from c_{i+1} and c_{i+2}. The pole $\mathbf{p}$ can be computed as the intersection of two lines l_i, l_{i+1} by checking if the intersection point is on the circle c_i.

Let $\mathbf{b}_0=[b_{0x} \quad b_{0y}]^{\mathrm{T}}$ be the first control point vector and $\mathbf{u}=[u_x \quad u_y]^{\mathrm{T}}$ be the unit vector parallel to $\mathbf{b}_1-\mathbf{b}_0$. Then the other control points of a typical class A Bézier curve are given by Eqn. (5.4). For control points $\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$, the pole $\mathbf{p}$ can be computed as

$$\mathbf{p}=[p_x/p_w \quad p_y/p_w]^{\mathrm{T}} \tag{6.3}$$

where

$$\begin{aligned} p_x &= b_{0x}+b_{0x}s^2+u_x-s((2b_{0x}+u_x)\cos\theta+u_y\sin\theta) \\ p_y &= b_{0y}+b_{0y}s^2+u_y-s((2b_{0y}+u_y)\cos\theta+u_x\sin\theta) \\ p_w &= 1+s^2-2s\cos\theta \end{aligned} \tag{6.4}$$

We can confirm that the computed pole $\mathbf{p}$ is on circle c_i, thus on c_{i+1} and c_{i+2}. Therefore, for cubic typical class A Bézier curves, we can compute the pole. For class A Bézier curves of degree 4, we can easily show that the triangles $\mathbf{pb}_2\mathbf{b}_3$ and $\mathbf{pb}_3\mathbf{b}_4$ are similar by using the relationship of $|\mathbf{b}_2-\mathbf{p}|/|\mathbf{b}_3-\mathbf{p}|=|\mathbf{b}_3-\mathbf{b}_2|/|\mathbf{b}_4-\mathbf{b}_3|$ and $\angle\mathbf{pb}_2\mathbf{b}_3=\angle\mathbf{pb}_3\mathbf{b}_4$. By repeating this process for class A Bézier curves of degree n, we can confirm that the triangles $\mathbf{pb}_i\mathbf{b}_{i+1}$ and $\mathbf{pb}_{i+1}\mathbf{b}_{i+2}(i>3)$ are similar. Thus for any typical class A Bézier curves of degree n, the pole $\mathbf{p}$ is represented by Eqn.(6.3).

For a typical class A Bézier curve of degree n, we can construct a logarithmic spiral that goes thorough all the control points. Since the control points converge to the Bézier curve itself as the degree is elevated, a typical class A Bézier curve converges to a logarithmic spiral. Thus typical class A Bézier curves can be considered as an approximation to logarithmic spiral.

7. INTERACTIVE CONTROL OF GENERAL CLASS A BÉZIER CURVES

This section presents a method for interactively controlling general class A Bézier curves. To generate a general class A Bézier curve, we start from a typical class A Bézier curve and then perturb the elements of the typical class A matrix such that endpoint constraints are satisfied.

We are given three point vectors $\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2$ and matrix perturbation parameters $\delta_0, \delta_1, \delta_2, \delta_3$. We would like to find a general class A Bézier curve whose endpoints are $\mathbf{a}_0, \mathbf{a}_2$ and their tangent vectors are parallel to $\mathbf{a}_1 - \mathbf{a}_0, \mathbf{a}_2 - \mathbf{a}_1$, respectively. A general class A matrix M_G can be obtained by perturbing a typical class A matrix M by $\delta_0, \delta_1, \delta_2, \delta_3$:

$$M_G = \begin{bmatrix} s\cdot\cos\theta + \delta_0 & s\cdot -\sin\theta + \delta_1 \\ s\cdot\sin\theta + \delta_2 & s\cdot\cos\theta + \delta_3 \end{bmatrix} \tag{7.1}$$

If $\delta_0, \delta_1, \delta_2, \delta_3$ are all 0 and M_G satisfies Eqn.(3.6), M_G becomes a typical class A matrix. However, if at least one of $\delta_0, \delta_1, \delta_2, \delta_3$ is not 0, the optimization process described in Section 5 does not work since changing only b_0 and T does not satisfy the endpoint constraints. We add θ as the optimization parameter and use an optimization function such that $\sum_{j=0}^{n-1} b_0 M_G^j \mathbf{u} = (\mathbf{a}_2 - \mathbf{a}_0)$ and the angle ϕ formed by $\mathbf{a}_1 - \mathbf{a}_2$ and $\mathbf{b}_{n-1} - \mathbf{b}_n$ becomes 0. See Fig. 6 for ϕ. We use an optimization process to find b_0, s, θ such that

$$f(s, b_0, \theta) = \left| \sum_{j=0}^{n-1} b_0 \cdot M_G^j \mathbf{u} - (\mathbf{a}_2 - \mathbf{a}_0) \right| + \phi$$

becomes 0. If M_G does not satisfy the class A conditions or the curvature is not verified with monotoncally varying, we do not draw the curve. Whether M_G generates curves with monotonically varying curvature or not depends on the positions of $\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2$ and the values of $\delta_0, \delta_1, \delta_2, \delta_3$.

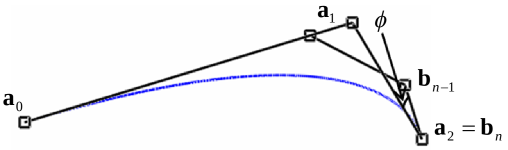
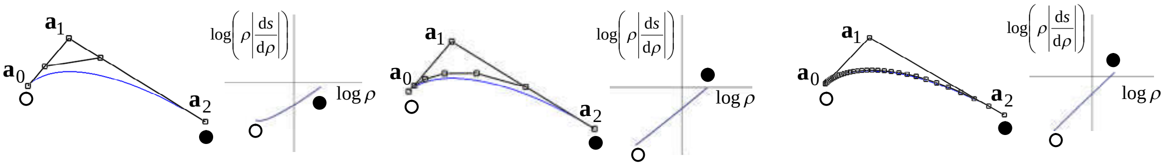


Fig. 6: Angle ϕ formed by $\mathbf{a}_1 - \mathbf{a}_2$ and $\mathbf{b}_{n-1} - \mathbf{b}_n$.

Once b_0, s, θ are found, we can easily generate the control points of the class A Bézier curve. However, the generated curve is not invariant under similarity transformations since the angle formed by $\mathbf{u}$ and $M_G\mathbf{u}$ differs depending on the direction of $\mathbf{u}$. To make the generated curve to be invariant under similarity transformations, we perform a similarity transformation of $\mathbf{a}_0$, $\mathbf{a}_1$ and $\mathbf{a}_2$ such that $\mathbf{u}$ becomes $[1 \quad 0]^T$ and $\mathbf{a}_2$ exists in the first quadrant. Then we perform the optimization process, compute the control points and transform them back.

8. RESULTS

Fig. 7 shows typical class A Bézier curves and their logarithmic curvature graphs. Fig. 7 (a) shows a cubic typical class A Bézier curves and its logarithmic curvature graph. We fit a line to its logarithmic curvature graph using the least square method. Its slope is 0.55 and its variance is 1.1097. Fig. 7 (b) and (c) are typical class A Bézier curves of degree 6 and 33, respectively. The slopes of the fitted straight lines of the logarithmic curvature graphs of Fig.7(b) and (c) are 0.80 and 0.97, and their variances are 0.1 and 4×10^{-6}, respectively. These results show how the curve segment gets closer to a logarithmic spiral whose slope of the logarithmic curvature graph is 1.

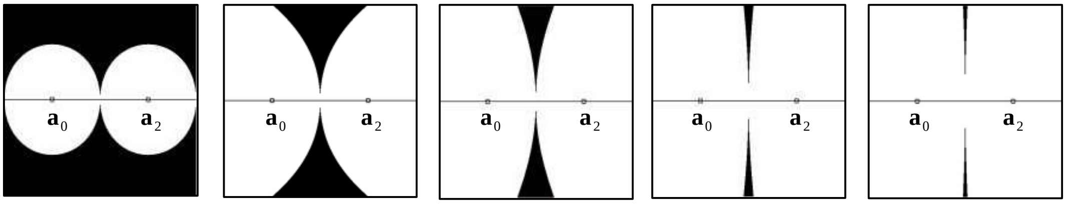


(a) degree 3 (b) degree 6 (c) degree 33

Fig. 7: Typical class A Bézier curves and their logarithmic curvature graphs.

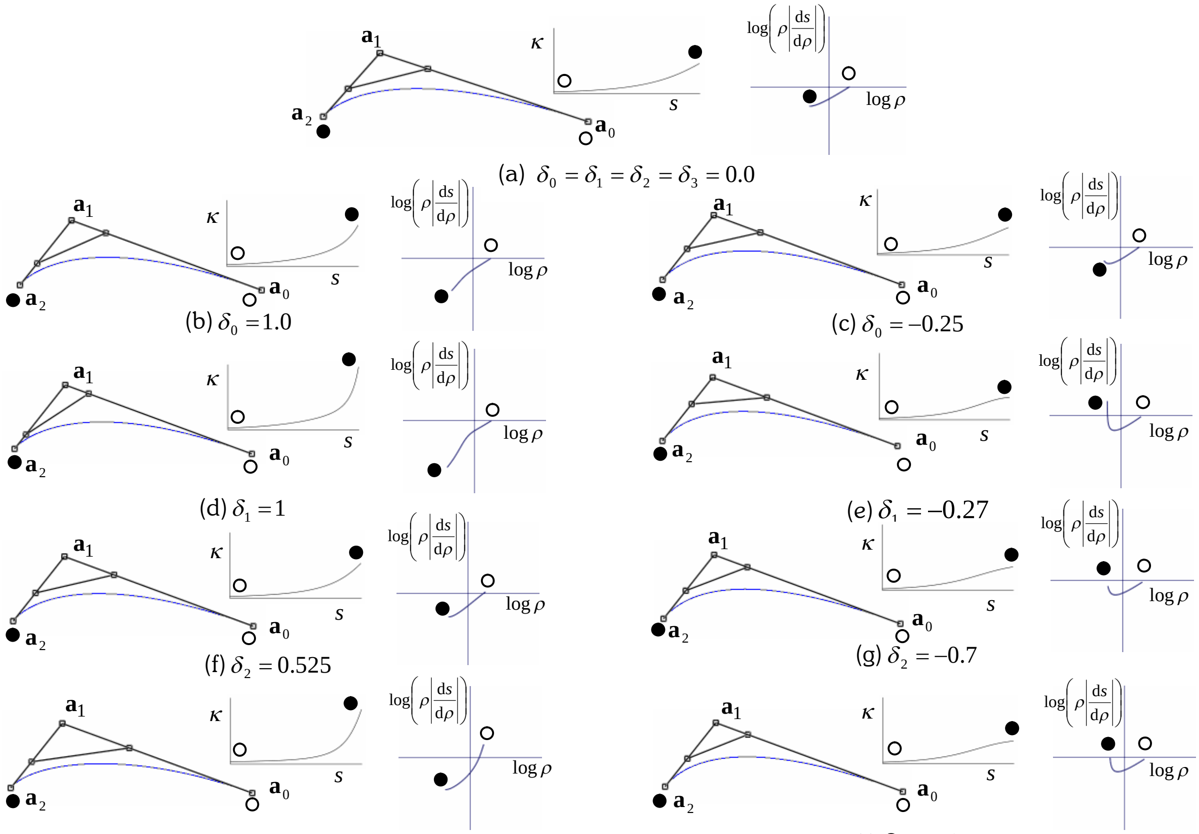
Typical class A Bézier curves are invariant under similarity transformations. To clarify if a curve segment is drawn depending on the positions of the three points for drawing a typical class A Bézier curve, we show the drawable regions in Fig. 8. We place the first point vector $\mathbf{a}_0$ and the third point vector $\mathbf{a}_2$ at $\begin{bmatrix}-1 & 0\end{bmatrix}^T$ and $\begin{bmatrix}0 & 1\end{bmatrix}^T$ respectively, and move $\mathbf{a}_1$ within the rectangle. If a typical class A Bézier curve is drawable (thus, the matrix satisfies the class A conditions and the curvature is monotonically varying), we draw the pixel corresponding to $\mathbf{a}_1$ with white. If not drawable, we draw the pixel with black. From Fig. 8, we can see that as the degree of typical class A Bézier curve is elevated, the drawable region gets larger.

Fig. 9 shows various kinds of cubic class A Bézier curves, their curvature plots and logarithmic curvature graphs. Fig. 9(a) is a typical class A Bézier curves since $\delta_0 = \delta_1 = \delta_2 = \delta_3 = 0.0$. Fig. 9 (b)-(i) shows various general class A Bézier curves whose δ_i are 0 except for the one indicated. Fig. 10 shows examples of general class A Bézier curves of degree 7 and 10. General class A Bézier curves of degree higher than 3 can also be generated in a similar manner.



(a) degree 3 (b) degree 4 (c) degree 5 (d) degree 20 (e) degree 30

Fig. 8: Drawable regions of typical class A Bézier curves.



(a) $\delta_0 = \delta_1 = \delta_2 = \delta_3 = 0.0$

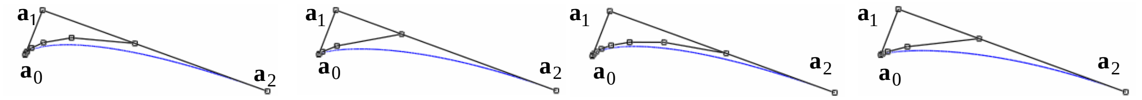
(b) $\delta_0 = 1.0$ (c) $\delta_0 = -0.25$

(d) $\delta_1 = 1$ (e) $\delta_1 = -0.27$

(f) $\delta_2 = 0.525$ (g) $\delta_2 = -0.7$

(h) $\delta_3 = 0.65$ (i) $\delta_3 = -0.3$

Fig. 9: General class A Bézier curves of degree 3 and their curvature plots and logarithmic curvature graphs.



(a) degree 7 $\delta_0 = \delta_1 = \delta_2 = \delta_3 = 0.0$
(b) degree 7 $\delta_0 = \delta_2 = 1.0, \delta_1 = \delta_3 = 0.5$
(c) degree 10 $\delta_0 = \delta_1 = \delta_2 = \delta_3 = 0.0$
(d) degree 10 $\delta_0 = \delta_2 = 1.0, \delta_1 = \delta_3 = 0.5$

Fig. 10: General class A Bezier curves of degree 7 and 10.

9. CONCLUSIONS

We have presented a method for interactively controlling planar class A Bézier curves. We first presented a method for interactively controlling typical class A Bézier curves. We showed that as the degree of a typical class A Bézier curve is elevated, the curve converges to a logarithmic spiral. Then we presented a method for controlling general class A Bézier curves by perturbing the coefficients of class A matrix such that endpoint constraints are satisfied. From the viewpoint of log-aesthetic curves, interactive control of a general class A Bézier curve segment means starting from an approximation to a log-aesthetic curve segment whose α is 1 (logarithmic spiral) and then loosening the linearity of the logarithmic curvature graph.

Future research includes investigating if the solution of the optimization process of typical and general class A Bezier curves is unique or not, clarifying how the shape of a general class A Bézier curve will change when δ_i is changed, and the extension to space curves. As pointed out by Cao's paper[1], more investigation is necessary for the class A condition of general matrices. Extending the idea of class A Bezier curves to rational Bezier curves, B-spline curves, or NURBS curves is also an interesting problem. For faring spline curves, a new faring algorithm that uses the linearity of the logarithmic curvature graph as an optimization constraint can be constructed.

10. REFERENCES

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