Computer-Aided Design and Applications

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Generating Optimal Path by Space-time Geometry

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ABSTRACT

The optimal path problem has many applications in engineering such as robotics motion planning and UAV path planning. A popular solution of the problem is to express the domain environment in terms of a graph and search for the optimal path between two nodes. This approach only includes the topological information of a path. The geometric information is omitted. This paper presents a new approach to generate an optimal path by using the space-time geometry. The environment is expressed in terms of index of refraction which determines the space-time geometry. Hence, both geometric and topological information of the environment are considered. A geodesic is computed on the space-time geometry to yield the optimal path.

Keywords: optimal path, geodesic, space-time geometry.
DOI: 10.3722/cadaps.2008.519-525

1. INTRODUCTION

The optimal path is basic to the solution of many problems in communications, navigation, robots, control, optimization, and enterprise planning. These are basically boundary value problems. The path is the geometric connection between two points. Instead of having a unique path, multiple paths may exist; hence, an optimal path is most desirable in such situation.

The *index of refraction* characterizes the velocity in a region. For example, on a highway, if the maximally allowed speed is c and if the instantaneous velocity of the vehicles is v, then that portion of the highway may be said to have an index of refraction $\eta = \frac{c}{\upsilon}$ for the "flow of traffic" at that time. In machining applications, the maximal feed rate of a machine tool may be limited to c while the instantaneous feed rate may be $\upsilon < c$; the portion being machined can be said to have a certain index of refraction.

It is well known that light seeks the most efficient path in space and in time since it travels along the geodesic of the space-time. The geometry of a geodesic is affected by the geometry of the space-time which can be featured by the spatial flow velocity in the space.

A unity index of refraction ($\eta = 1$) implies a constant velocity field which gives a flat space-time with zero Gaussian curvature. As a result, the geodesic on the space-time geometry is a straight line. When the index of refraction is not one ($\eta \neq 1$), then the space-time geometry is non-flat and results a curved geodesic. This paper presents an approach to generate an optimal path by modelling the space-time geometry, computing the geodesic and projecting it onto the space.

2. RELATED WORKS

Two major applications of path planning are for robotics[1-3] and unmanned aerial vehicles[4-6] (UAV). Both refer to seek a path between two points in a domain with some no entry regions. A popular approach is to use a graph to represent the environment of these no entry regions which can be obstacles and/or threat regions. A graph search algorithm such as Dijkstra algorithm, breath-first search, A* search algorithm[7] are commonly employed to find the optimal path. The path usually makes a detour around the no entry regions. Efficiency and effectiveness are the major concerns in this approach. Furthermore, the graph just describes the topology (possible connections) of the environment which is considered as discrete. Fast marching algorithm[8] is similar to Dijkstra's algorithm but it is consistent in a continuous domain.

In order to have the detail geometry of path such as the curvature and distance, the geometry of the environment must be considered. This paper presents a graphical approach for solving a boundary value problem and yielding an optimal solution. The environment of the domain is expressed in term of the index of refraction instead of a graph. The space-time geometry embedded with the index of refraction is modeled. An optimal path between two given points subject to various indexes of refraction due to the environment is obtained by projecting the geodesic onto the space.

3. SPACE-TIME GEOMETRY WITH UNITY INDEX OF REFRACTION

Consider a particle moving in a region R with constant velocity υ. At time t, the position of the particle is $\left(u_1 = u_1(t-t_0), u_2 = u_2(t-t_0)\right)$, hence the distance traveled and the time of flight is $r=\sqrt{u_1^2+u_2^2}$ and $t-t_0=\frac{r}{\upsilon}$ respectively (where t_0 is the delay). Rearranging the time of flight yields $\upsilon(t-t_0)=r$. The geometry of the space (*r*) and time (*t*) is represented by an inverted cone K with a half angle of $\frac{\pi}{4}$ as depicted in figure 1. Under constant velocity condition, the particle is traveling along a geodesic on the space-time geometry. Projecting this geodesic onto the space gives the optimal (in time) path. Since the velocity is constant, the Gaussian curvature of the cone is zero and the space-time geometry is flat.

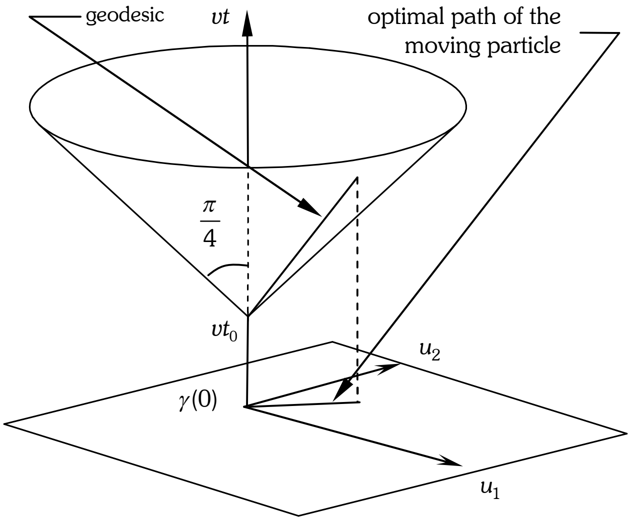

Fig. 1: Flat space-time geometry with $\eta = 1$.

4. TWO REGIONS WITH NON-UNITY INDEX OF REFRACTION

When the original point $\mathbf{p}_o$ and the target point $\mathbf{p}_t$ are located in two regions R_1 and R_2 with different constant velocities υ_1 and υ_2 respectively. Assuming that $t_0 = 0$, then time of flight is rewritten as

$$t = \frac{r_1 + \eta \cdot r_2}{v_1} \tag{1}$$

where $\eta = \frac{v_1}{v_2}$ is the index of refraction; r_1 and r_2 are the distance traveled by the particle in region R_1 and R_2 respectively.

Let the intersection of the path and interface of the two regions be an intermediate point $\mathbf{p}_i$, then point $\mathbf{p}_i$ is the *secondary* target point of the path section in region R_1 and it is also the *secondary* original point of the path section in region R_2. Therefore, cone K_1 with half angle of $\frac{\pi}{4}$ and vertex at point $\mathbf{p}_o$; and cone K_2 with half angle of θ (where $\tan\theta = \eta$) and vertex at $\mathbf{p}_i$ are used to represent the space-time geometry as shown in figure 2. Note that the vertex of the cone K_2 representing the space-time in region R_2 is *not* on point $\mathbf{p}_i$. It has a delay for the particle to travel from the original point $\mathbf{p}_o$ to the secondary target point $\mathbf{p}_i$ at the interface.

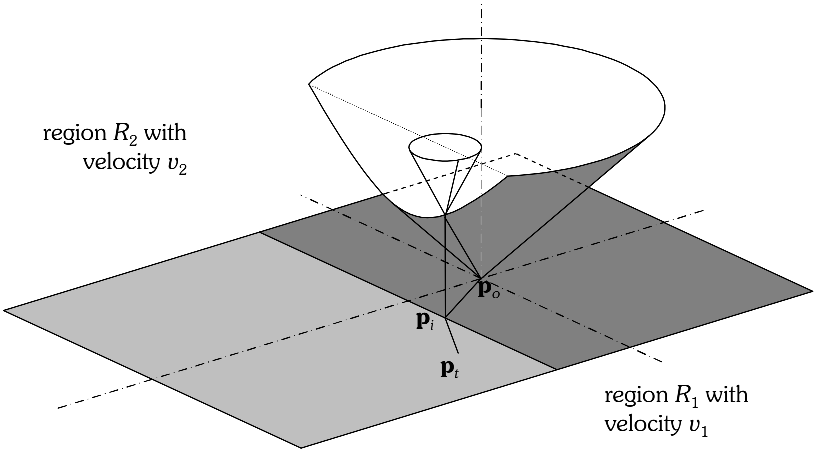

Fig. 2: The particle path across regions with index of refraction.

Hence, a complete space-time plot for two regions with an index of refraction η can be obtained by sweeping the cone K_2 along the intersection curve c_{12} between the cone K_1 and the region interface as depicted in Figure 3.

The geometry of the space-time provides an optimal solution to a boundary value problem of the following kind: *finding the path with minimum time of flight between the original point in region R_1 and the target point in the other region R_2.*

The first part of the problem is easy. Since the Gaussian curvature of a cone is zero, the solution is simply straight line connecting the original point to the points on the interface. The second half is slightly harder: it involves traversing in a direction in the other region.

With the given locations of the original $\mathbf{p}_o$ and the target point $\mathbf{p}_t$, the approach is to trace from the target point $\mathbf{p}_t$ back to the original point $\mathbf{p}_o$ along the geodesic on the space-time geometry. Starting from the target point $\mathbf{p}_t$, project it onto the space-time geometry to obtain point $\mathbf{q}_t$ as shown in figure 4. The point $\mathbf{q}_i$ can be found along the geodesic on the cone K_2 from the point $\mathbf{q}_t$. The path is the projection of the geodesic $\mathbf{p}_0\mathbf{q}_i$ and $\mathbf{q}_i\mathbf{q}_t$ on both cones onto the plane (u_1, u_2).

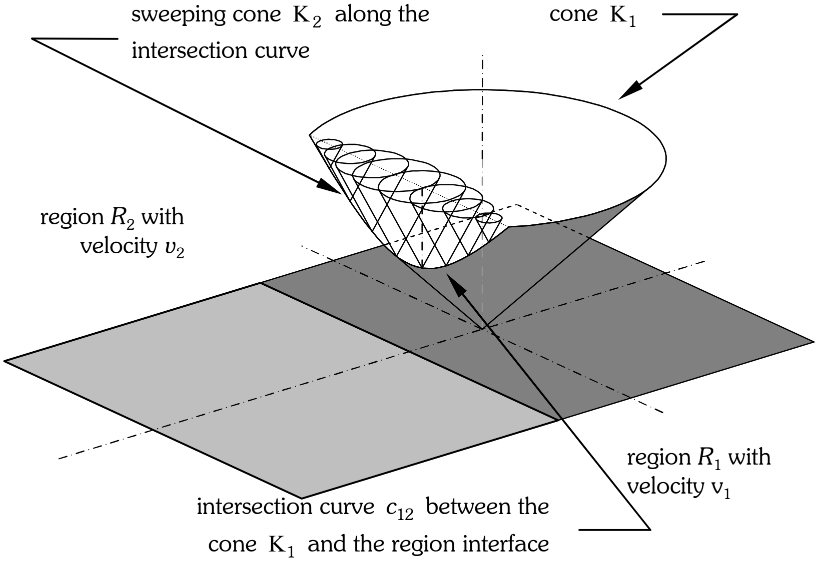

Fig. 3: The space of two regions with straight boundary at the interface.

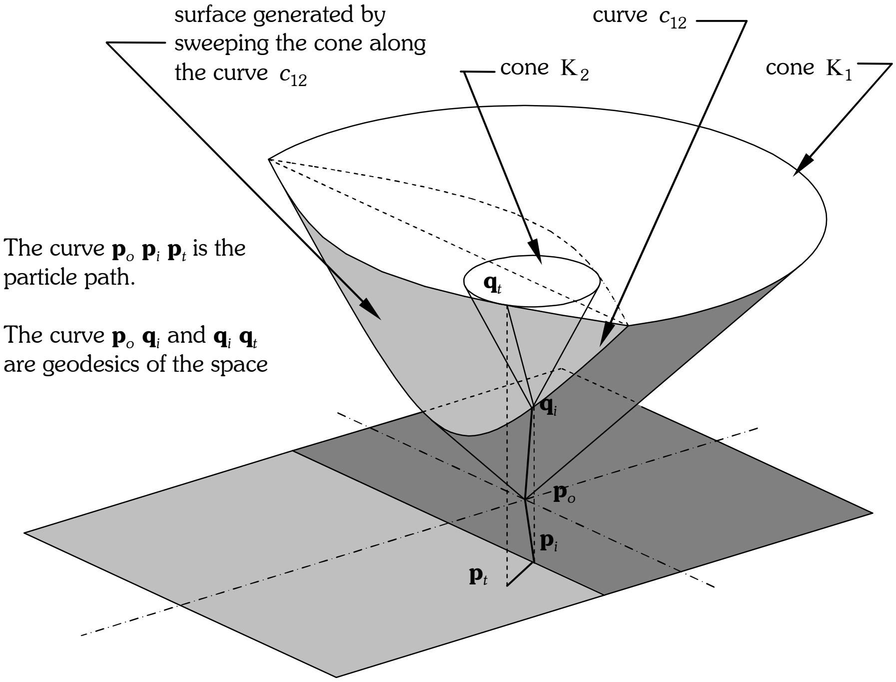

Fig. 4: Graphical solution for the boundary value problem.

5. APPLICATION EXAMPLE – PATH PLANNING FOR UAV

Path planning for an unmanned aerial vehicle (UAV) is a typical boundary value problem since it involves the path between two points: original point and target point. A region R is characterized by a spatially dependent threat parameter per unit distance traveled by the aircraft. The threat is mainly due to the adversarial equipment such as radar. In such situation, the parameter refers to the rate of detection of the aircraft (UAV) per unit length of the path.

Once the aircraft is detected, the radar initiates tracking and, after some response time, launches a missile. Tracking requires continuous observation of the aircraft during a response time interval. If the number of detection is too low, then the track is lost. The aircraft must be reacquired and new track must be initiated. Hence, the aircraft will be safe after entering the threat region provided the time interval is short enough.

Hence, the problem is formulated as to search for a path γ in a region R_j with rate of detection λ_j per unit distance along the path from the original position $\mathbf{p}_o$ to the target position $\mathbf{p}_t$ such that the time of flight is minimum.

Consider an aircraft traveling with velocity υ from the original point $\mathbf{p}_o$ in region R_1 with threat parameter λ_1 to the target point $\mathbf{p}_t$ in region R_2 with threat parameter λ_2, the apparent velocity of the aircraft in these two regions are $\frac{\upsilon}{\lambda_1}$ and $\frac{\upsilon}{\lambda_2}$ respectively. The index of refraction is expressed as $\eta = \frac{\upsilon_1}{\upsilon_2}$

$$\eta = \frac{\upsilon_1}{\upsilon_2} = \frac{\lambda_2}{\lambda_1} \tag{2}$$

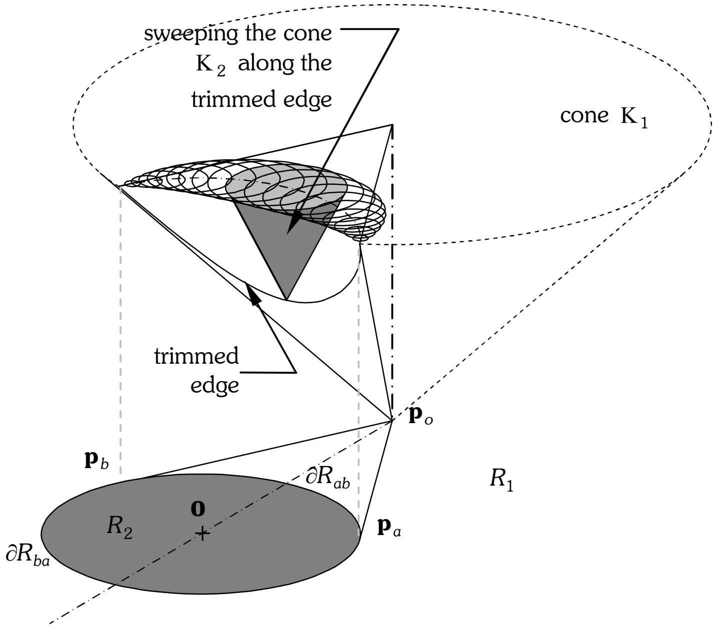

Fig. 5: The space-time of two regions with circular boundary at the interface.

In many cases, the threat region is finite and has a closed boundary at the interface. In figure 5, $\mathbf{p}_o$ is the original point in region R_1 with threat parameter λ_1. An adversarial radar is located at point $\mathbf{o}$ which covers a circular region R_2 with threat parameter $\lambda_2 (> \lambda_1)$. There are two extreme points $\mathbf{p}_a$ and $\mathbf{p}_b$ for the aircraft to enter region R_2 such that $\mathbf{p}_o\mathbf{p}_a$ and $\mathbf{p}_o\mathbf{p}_b$ are tangent to the boundary of region R_2. These two points divides the boundary into two parts: ∂R_{ab} and ∂R_{ba} where ∂R_{ab} consists of a set of intermediate points. The space-time of region R_1 is represented by a cone K_1 with vertex angle of $\frac{\pi}{4}$. With the introduction of region R_2, the cone K_1 is trimmed by the boundary ∂R_{ab} and the space-time of region R_2 is obtained by sweeping a cone K_2 with vertex angle of θ (where $\tan\theta = \eta$) along the trimmed edge of the cone K_1.

Figure 6(a) shows the space-time geometry within the circular region R_2. For any point $\mathbf{p}$ in region R_2, there exists another point $\mathbf{q}$ which is the projection of point $\mathbf{p}$ along the vt axis onto the space-time geometry. This point $\mathbf{q}$ lies on the surface of a cone K_2 with the vertex $\mathbf{q}_i$ at the trimmed edge. The line $\mathbf{p}_0\mathbf{q}_i$ is a geodesic on the cone K_1. The optimal path from the original point $\mathbf{p}_0$ to point $\mathbf{p}$ is obtained by projecting the geodesic $\mathbf{p}_0\mathbf{q}_i$ and $\mathbf{q}_i\mathbf{q}$ onto the space (u_1,u_2). Figure 6(b) plots the optimal paths from the points in region R_2 to original point $\mathbf{p}_0$ in region R_1. Note that there are two (white) zones in region R_2 such that their corresponding space-time geometries are two conical surfaces with vertex at point $\mathbf{q}_a$ and point $\mathbf{q}_b$ on the trimmed edge. This implies that point $\mathbf{p}_a$ and $\mathbf{p}_b$ are the entrances for the optimal paths to enter these two zones respectively.

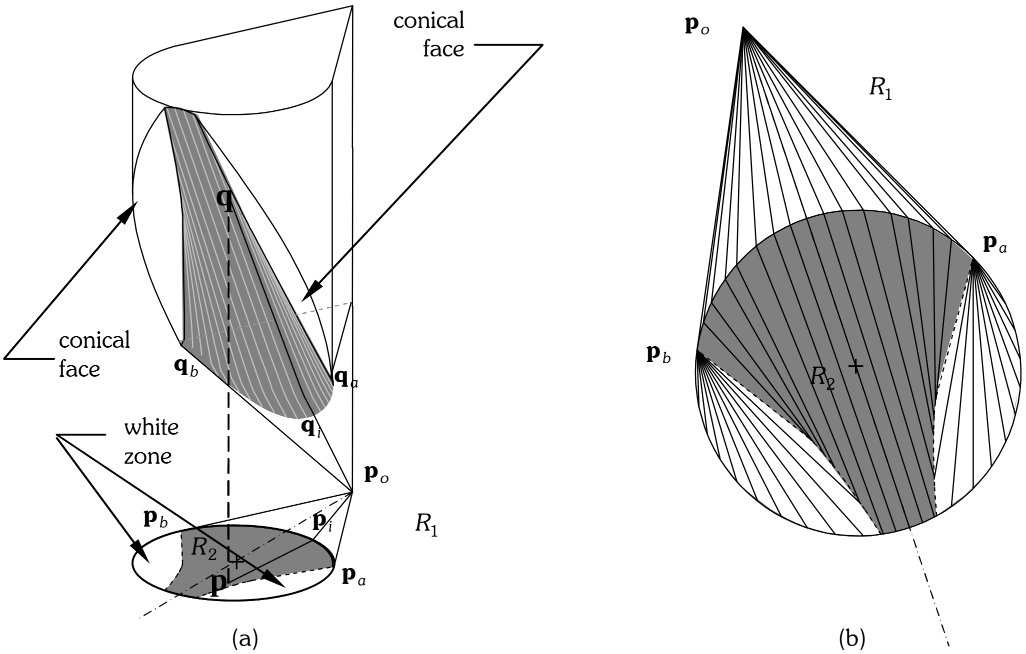

Fig. 6: The optimal paths from point $\mathbf{p}_0$ to the points in region R_2.

A mission considered here is to fly from an original point $\mathbf{p}_0$ to a target point $\mathbf{p}_t$ in the region R_1 through a *waypoint* $\mathbf{p}$ in the region R_2. The region R_2 is guarded by radar at point **o.** The threat parameters for the region R_1 and R_2 are λ_1 and λ_2 $(>\lambda_1)$ respectively. The path planning is to seek an optimal path so that the mission is completed. Since the paths between two points are reversible, two families of optimal paths are developed with respect to the original point $\mathbf{p}_0$ and the target point $\mathbf{p}_t$. Hence, every point within the region R_2 corresponds to an intersection of two optimal paths from the point $\mathbf{p}_0$ and the point $\mathbf{p}_t$ respectively as depicted in figure 7. Therefore, for every waypoint $\mathbf{p}$ in the region R_2, there exists two optimal paths from the points $\mathbf{p}_0$ and $\mathbf{p}_t$ in the region R_1 such that the combined path $\mathbf{p}_t$ $\mathbf{p}$ $\mathbf{p}_t$ is the optimal for the mission.

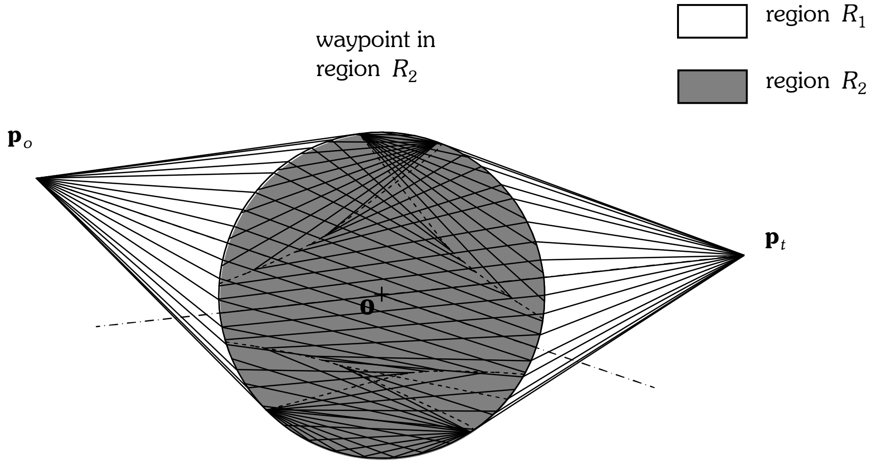

Fig. 7: The paths from original point to target point through a waypoint in region R_2.

6. DISSCUSSION AND SUMMARY

The UAV path generated by this approach is optimum in time. However, the radar cross section of an aircraft has not been counted. Furthermore; the safety is not directly considered. Suppose the path consists of *n* segments, then the theoretical number of detections at time *t* in region R_j is $\mu_j = \lambda_j \cdot r_j \cdot (t - t_{j-1})$ where t_{j-1} is the time when the aircraft crosses the interface between the regions R_{j-1} and R_j, and r_j is the distance traveled in the region R_j. Let the probability of detection at time *t* in region R_j be $P(\mu_j)$, then the aircraft will be detected at time *t* if $P(\mu_j)$ is high enough (for example $P(\mu_j)$ >0.8). Hence, the velocity of the aircraft is important. It must be maintained above a minimum value so that the traveling time in a threat region is short. Hence, the probability of detection is low. In fact, even the aircraft is detected; it is still safe provided it leaves the threat region within a time interval less than the missile launching time.

The notion of the optimal path, in space and/or in time, is of interest to UAV path planning under the situation of multiple regions with different threat parameters. Likewise, for machining when the material is not homogeneous or isotropic and for motion planning in robotics, in the presence of other active agents, "traffic" manifests. The index of refraction between two adjacent regions implies the existence of forces which distorts the space-time. Importantly, the space under consideration is often not "flat" or Euclidean. It is the hope that CAD/CAM researchers will find the techniques developed in this paper useful, in particular non-Euclidean domains and its applications fruitful.

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