# Non-stationarization of the Typical Curves and its Extension to Surfaces

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**ABSTRACT**

It is known that if the degree of the typical plane Bézier curve is increased infinitely, the curve will converge to the logarithmic (equiangular) spiral. The logarithmic spiral is one of log-aesthetic curves and they are formulated by $\alpha$: the slope of the logarithmic curvature graph. In this paper we define the non-stationary typical Bézier curve by making the transition matrix of the typical Bézier curve non-stationary and dependent on each side of the control polyline and defining the transition matrix in the Frenet frame. We propose a method that generates a curve such that if its degree is increased infinitely it will converge to a log-aesthetic curve with arbitrary $\alpha$ and $\beta$: the slope of the logarithmic torsion graph in case of the space curve, by controlling the relationship between the rotation angle and the scaling factor. Furthermore we extend the non-stationarization for free-form surfaces and propose the non-stationary typical surface with the unit scaling factor.

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## 1 INTRODUCTION

Yoshida et al. proved that the typical plane Class A Bézier curve [6] converges to a logarithmic (equiangular) spiral when its degree is increased infinitely [12]. The logarithmic spiral is one of the typical examples of the log-aesthetic curves and they are formulated by use of the slope of the log-aesthetic curve $\alpha$. The relationship between the typical plane Class A Bézier curve and $\alpha$ has not been cleared.

Hence in order to define a curve which converges to a log-aesthetic curve when its degree is increased infinitely, we do not fix the rotation angle and the scaling factor of the transition matrix used to define the typical Bézier curve and make it non-stationary. By making the transition matrix depend on each side of the control polyline, we define the non-stationary typical Bézier curve. We will propose a Bézier curve of degree n whose rotation angle is controlled with a fixed scaling factor, or whose scaling factor is controlled with a fixed rotation angle and which converges to a log-aesthetic curve with an arbitrary $\alpha$ value when its degree is increased infinitely. The control points of the proposed Bézier curve of degree n are defined to converge to a log-aesthetic curve. If they are considered to be those of a B-spline curve of degree $m$ $(\leq n)$, the curve converged to the same log-aesthetic curve when the number of its control points, i.e. that of its segments is increased infinitely.

We will make clear the conditions for monotonicity of curvature of the B-spline curve, especially in case of $m=3$.

The radius of torsion as well as that of curvature of the log-aesthetic space curve is given by linear functions of the arc length and the logarithmic torsion graph is given by a straight line whose slope is a contant [9]. In this paper, we will propose a space curve which has arbitrary $\alpha$ and $\beta$ as a non-stationary typical Bézier curve generated by transition matrices defined in the Frenet (moving) frame. Furthermore we will propose the non-stationary typical surface whose sides of its control mesh have a constant length.

## 2 RELATED WORK

In this section, we will review the Class A Bézier curve and the log-aesthetic curve, especially for the latter we will explain the general equations of aesthetic curve and their parametric representations.

### 2.1 Class A Bézier Curve

The Class A Bézier curve is a curve whose curvature and torsion are monotonic proposed by Farin [6]. Let $b_i$ $(0 \le i \le n)$ be control points of a Bézier curve of degree $n$ and $\Delta b_j = b_{j+1} - b_j$ $(0 \le j \le n-1)$. We assume that there is a transition matrix $M$ such that $\Delta b_j = M^j \Delta b_0$. If the matrix $M$ satisfies the following equation for $t \in [0,1]$ and an arbitrary vector $v$ such that $|v| = 1$,

$$|(1-t)v + tMv| \ge |v| \tag{2.1}$$

the curve defined by the control points $b_i$ is a Class A Bézier curve which curvature and torsion is monotonic. Eqn. (2.1) means that for any $v$ such that $|v|=1$ the line segment defined by $v$ and $Mv$ does not intersect with the unit sphere except for its end point. The matrix $M$ satisfies the following two conditions.

1. The angle between $v$ and $Mv$ must be smaller than 90 degrees.
2. The matrix $M$ must map an arbitrary point on the unit sphere to a point outside the sphere.

Farin showed that to satisfy Condition 1, the eigenvalues of the following two matrices should be non-negative,

$$M^T + M - 2I, \quad M^T M - I \tag{2.2}$$

where $I$ is an identity matrix. To satisfy Condition 2, the singular values of $M$: $\sigma_1$, $\sigma_2$ $(\sigma_1 > \sigma_2)$ are equal to 1 or greater. To prove curvature monotonicity, Cao and Wang [4] gave the following conditions for $\sigma_1$ and $\sigma_2$ of a symmetric matrix,

$$2\sigma_1 \ge \sigma_2 + 1, \quad 2\sigma_2 \ge \sigma_1 + 1 \tag{2.3}$$

Hence in order for a symmetric matrix $M$ to produce a Class A Bézier curve, $M$ in Eqn. (2.1) must have non-negative eigenvalues and its singular values $\sigma_1$ and $\sigma_2$ must be equal to or greater than 1. Furthermore it must satisfy Eqn. (2.3). These conditions on $M$ are called the Class A conditions.

Farin gave the typical Class A matrix as an example of Class A matrices [6]. The Bézier curve generated by the typical Class A matrix is identical to the typical curve proposed by Mineur et al. [10] based on the research carried out by Higashi et al [7]. If the transition matrix $M$ consists of a rotation angle $\theta_r < \pi/2$ and a scaling factor $s$ and the following enequality is satisfied,

$$\cos\theta_r > \frac{1}{s_f} \text{ (if } s_f > 1\text{) or } \cos\theta_r > s_f \text{ (if } s_f < 1\text{)} \tag{2.4}$$

the matrix $M$ is Class A.

### 2.2 Log-aesthetic Curve

This section discusses several important properties of log-aesthetic curves. Note that an aesthetic curve is a curve whose logarithmic curvature graph is given by a straight line.

### 2.2.1 General Equations of Aesthetic Curves

For a given curve, we assume the arc length of the curve and the radius of curvature are denoted by $s$ and $\rho$, respectively. The horizontal axis of the logarithmic curvature graph measures $\boldsymbol{log}\,\rho$ and the vertical axis measures $\boldsymbol{log}(ds/d(\boldsymbol{log}\,\rho)) = \boldsymbol{log}(\rho\, ds/d\rho)$. If the LCG is given by a straight line, there exists a constant $\alpha$ such that the following equation is satisfied:

$$\boldsymbol{log}\left(\rho \frac{ds}{d\rho}\right) = \alpha\, \boldsymbol{log}\,\rho + C \tag{2.5}$$

where $C$ is a constant. The above equation is called the fundamental equation of aesthetic curves [8]. Rewriting Eqn. (2.5), we obtain:

$$\frac{1}{\rho^{\alpha-1}}\frac{ds}{d\rho} = e^{C} = C_0 \tag{2.6}$$

Hence there is some constant $c_0$ such that:

$$\rho^{\alpha-1}\frac{d\rho}{ds} = c_0 \tag{2.7}$$

From the above equation, when $\alpha \neq 0$, the first general equation of aesthetic curves

$$\rho^{\alpha} = c_0 s + c_1 \tag{2.8}$$

is obtained. If $\alpha = 0$, we obtain the second general equation of aesthetic curves aesthetic curves

$$\rho = c_0 e^{c_1 s} \tag{2.9}$$

The curve which satisfies Eqn. (2.8) or Eqn. (2.9) is called the log-aesthetic curve.

### 2.2.2 Parametric Expressions log-aesthetic Curves

In this subsection, we will show parametric expressions of the log-aesthetic curves.
We assume that a curve $C(s)$ satisfies Eqn. (2.8). Then

$$\rho(s) = (c_0 s + c_1)^{\frac{1}{\alpha}} \tag{2.10}$$

As $S$ is the arc length, $|dC(s)/ds| = 1$ (refer to, for example, [5]) and there exists $\theta(s)$ satisfying the following two equations:

$$\frac{dx}{ds} = \boldsymbol{cos}\,\theta, \quad \frac{dy}{ds} = \boldsymbol{sin}\,\theta \tag{2.11}$$

Since $1/(d\theta/ds)$,

$$\frac{d\theta}{ds} = (c_0 s + c_1)^{-\frac{1}{\alpha}} \tag{2.12}$$

If $\alpha \neq 1$,

$$\theta = \frac{\alpha (c_0 s + c_1)^{\frac{\alpha-1}{\alpha}}}{(\alpha-1)c_0} + c_2 \tag{2.13}$$

If the start point of the curve is given by $P_0 = C(0)$,

$$C(s) = P_0 + e^{ic_2}\int_0^s e^{i\frac{\alpha(c_0 u + c_1)^{\frac{\alpha-1}{\alpha}}}{(\alpha-1)c_0}}\, du\theta \tag{2.14}$$

For the second general equation of aesthetic curves expressed by Eqn. (2.9),

$$\frac{d\theta}{ds} = \frac{1}{c_0}e^{-c_1 s} \tag{2.15}$$

$$\theta = -\frac{1}{c_0 c_1}e^{-c_1 s} + c_2 \tag{2.16}$$

Therefore the curve is given by

$$C(s) = P_0 + e^{ic_2} \int_0^s e^{-\frac{i}{c_0 c_c} e^{-c_1 u}} du \tag{2.17}$$

## 3 APPROXIMATION OF LOG-AESTHETIC CURVE

The log-aesthetic curve is represented in an integral form as explained in the previous section and it is necessary to perform numerical integration to calculate a point on the curve except for $\alpha = 1$ and $2$. If $\alpha = 1$ or $2$, its form can be integrated analytically. For aesthetic curve design, the method to specify the start and end points and sometimes also tangent vectors there is usually adopted and to generate a curve satisfying these conditions, it is necessary to search $c_0$ and $c_1$ in Eqn. (2.8) numerically. Especially to make a specific point to be the end of the curve, it is necessary to calculate the end point of the curve for given $c_0$ and $c_1$ repeatedly [1, 2, 11]. If we can approximate a log-aesthetic curve by a Bézier curve or B-spline curve, we can omit numerical integration and accelerate parameter search.

Here we think about the approximation of a log-aesthetic plane curve by a Bézier curve. We assume that the slope of the logarithmic curvature graph $\alpha$ is not equal to $0$ or $1$ and the start point of a log-aesthetic curve in Eqn. (2.14) is at the origin and its tangent vector there is the same as the positive direction of the $x$-axis. We distinguish similar figures and simplify Eqn. (2.14) as follows:

$$C(s) = \int_0^s e^{iau^{\frac{\alpha-1}{\alpha}}} du \tag{3.1}$$

By the above simplification, if we use an adequate integration range, we can express an arbitrary log-aesthetic curve with $\alpha \neq 0, 1$.

As we mentioned in the previous section, the transition matrix $M$ of a typical curve consists of a rotation whose angle is $\theta_r$ and a scaling whose factor is $s_f$. Note that if $M$ is made dependent to each side of the control polyline, i.e. be non-stationarized, the control points of an arbitrary Bézier curve can be specified. To improve the controllability of the curve, we do not change both $\theta_r$ μ and $s_f$ at the same time. We fix $\theta_r$ and connect two consecutive sides of the control polyline at a constant angle and change $s_f$, or we fix $s_f$ s as 1 and use sides with the same length and control angles between two consecutive sides. In the following discussions, we propose these two methods to approximate the log-aesthetic curve.

### 3.1 Scaling Factor $s_f = 1$

We perform integration in Eqn. (2.18) discretely as follows:

$$C(s) \approx \sum_{j=0}^{n-1} e^{ia(j\Delta s)^{\frac{\alpha-1}{\alpha}}} \Delta s \tag{3.2}$$

where the total length of the curve is assumed to be $n\Delta s$. The origin is considered to be the first control point and $\Delta s$ is considered to be the distance between two control points, the above equation specifies $n+1$ control points sequentially each of which is apart from its previous control point by $\Delta s$. The direction angle (angle to the positive direction of the $x$-axis) of the $j+1$-th side of the control polyline is given by $a(j\Delta s)^{\frac{\alpha-1}{\alpha}}$ and the difference of the direction angles $\Delta\theta_j$ between two tangent vectors at $(j+1)\Delta s$ and $(j+2)\Delta s$ is given by

$$\Delta\theta_j = a\{(j+2)\Delta s\}^{\frac{\alpha-1}{\alpha}} - a\{(j+1)\Delta s\}^{\frac{\alpha-1}{\alpha}} \tag{3.3}$$

Hence if $\Delta s$ is kept constant, Eqn. (3.2) is similar to the representation of the Class A Bézier defined by a rotation matrix $M_j = R_{\Delta\theta_j}$ whose scaling factor and rotation angle are equal to 1 and $\Delta\theta_j$, respectively.

The matrix $M_j$ of the usual Cass A Bezier curve does not depend on $j$ and is constant, but in this formulation $M_j$ does depend on $j$. Namely let the control points of a Bézier curve of degree $n$ be $b_i$ ($0 \le i \le n$) and $\Delta b_j$ be $b_{j+1} - b_j$ ($0 \le j \le n-1$). For given matrices $M_j$ ($j = 0, \cdots n-2$), they satisfy $\Delta b_j = M_{j-1} \cdots M_0 \Delta b_0$. Hence we call this type of curve the non-stationary typical Bézier curve. Since the end point of the curve coincides with the last control point, it is not necessary to carry out numerical integration. The total rotation angle from the start point to the end point is given by $\{(n-1)\Delta s\}^{\frac{\alpha-1}{\alpha}}$.
The positions of the control points are determined by Eqn. (3.2). If $\Delta s$ can be assumed to be very small, the ratio of the rotation angles is concisely expressed. If $\Delta s$ is small enough, by taking the first term of the Taylor expansion, $\Delta\theta_j$ is given by

$$\Delta\theta_j = a\frac{\alpha-1}{\alpha}\{(j+1)\Delta s\}^{-\frac{1}{\alpha}}\Delta s \tag{3.4}$$

Therefore the ratio of two consecutive rotation angles $\Delta\theta_{j+1} / \Delta\theta_j$ is given by

$$\frac{\Delta\theta_{j+1}}{\Delta\theta_j} = \left(\frac{j+1}{j+2}\right)^{\frac{1}{\alpha}} \tag{3.5}$$

The top of Fig.1 shows an example of the Bézier curve defined by the method discussed above. The $\alpha$ value in Eqn. (3.2) to generate this curve is $-0.5$. The curve is rendered as a green pipe and each of its control points is rendered as a red sphere. The control points are connected with red pipes. The bottom of Fig.1 shows curvature and logarithmic curvature graphs of the curves of degrees $99$, $49$ and $24$, i.e. the number of the control points of these curves are changed from $100$ to $50$, to $25$. The $\alpha$ value of these curves is specified to be $-0.5$. When the number of the control points is $100$, its LCG is given by almost a straight line and its slope is about $-0.54$, which is close to the specfied value $-0.5$. As the degree is decreased, although the monotonicity of their curvature is kept, their LCGs deviate from a straight line.

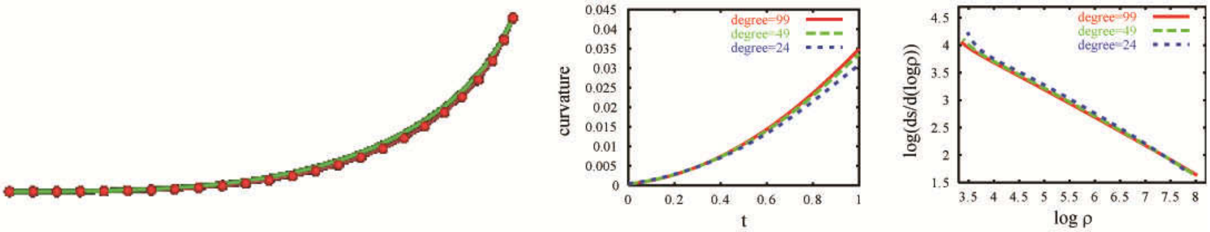


Fig. 1: Left: approximate Bézier curve of log-aesthetic plane curve (degree 24, $\alpha = -0.5$), Right: Comparison of curvature and LCG of curves of different degrees.

In order to perform numerical integration of Eqn. (3.1) more precisely, the trapezoidal rule or Simpson's rule should be used. For example, if the trapezoidal rule is adopted, the control points should be generated by halving the lengths of the first and last sides. However there are cases where curvature monotonicity and linearity of the LCG are strongly broken. Although we restrict our discussions within the case of the cubic B-spline curve, we will explain the relationship between the length of the sides of the control polyline and curvature at the end points in subsection 3.4.

## 3.2 Constant Rotation Angle

Although in Eq.(3.4) the displacement of the arc length $\Delta s$ is kept constant, we fix the rotation angle $\Delta\theta_j$ and make it $\Delta\theta$ which is independent from $j$. Then

$$\Delta\theta = a\frac{\alpha-1}{\alpha}\{(j+1)\Delta s_j\}^{-\frac{1}{\alpha}}\Delta s_j \tag{3.6}$$

Hence

$$\Delta s_j = (j+1)^{\frac{1}{\alpha-1}}\left\{\frac{\alpha\Delta\theta}{a(\alpha-1)}\right\}^{\frac{1}{\alpha-1}} \tag{3.7}$$

$\Delta s_j$ is independent from $j$ and the scaling factor is given by

$$\frac{\Delta s_{j+1}}{\Delta s_j} = \left(\frac{j+2}{j+1}\right)^{\frac{1}{\alpha-1}} \tag{3.8}$$

The curve becomes a non-stationary typical Bézier curve defined by $M_j = s_j R_{\Delta\theta}$ with a constant rotation angle $\Delta\theta$.

## 3.3 Approximation Examples

In this subsection, we apply our approximation method to the logarithmic spiral and the clothoid curve.

### *3.3.1 Logarithmic Spiral*

By use of a parameter $t$ which determines the direction angle, the general equation of the logarithmic spiral is given by

$$C(t) = e^{(a+ib)t} \tag{3.9}$$

By differentiating the above equation, we obtain

$$\frac{dC(t)}{dt} = (a+ib)e^{(a+ib)t} \tag{3.10}$$

Here we keep the change of the rotation angle $\Delta\theta = b\Delta t$ constant. The scaling factor, i.e. the ration of the change of the arc length is $\Delta s_{j+1}/\Delta s_j = e^{a(j+1)\Delta\theta/b}/e^{aj\Delta\theta/b}$. It is $e^{a\Delta\theta/b}$ and constant. Hence the matrix $M_j$ does not depend on $j$ and the obtained curve is a usual Class A Bézier curve defined by a stationary matrix. This fact indicates the property shown by Yoshida et al. that when the degree $n$ of a Class A Bézier curve is increased, it converges to a log-aesthetic curve.

### *3.3.2 Clothoid Curve*

Since $\alpha$ of the clothoid curve is equal to $-1$, if the change of the arc length is constant,

$$\Delta\theta_j = a^2(j+1)\Delta s^2 \tag{3.11}$$

If $\Delta s$ is small enough, the ratio of the tow consecutive rotation angles is given by

$$\frac{\Delta\theta_{j+1}}{\Delta\theta_j} = \left(\frac{j+2}{j+1}\right)^{\frac{1}{2}} \tag{3.12}$$

If the change of the angle is constant,

$$\Delta s_j = (j+1)^{-\frac{1}{2}}\left(\frac{1}{2a}\Delta\theta\right)^{\frac{1}{2}} \tag{3.13}$$

If $\Delta s_j$ is small enough, the scaling factor is given by

$$\frac{\Delta s_{j+1}}{\Delta s_j} = \left(\frac{j+1}{j+2}\right)^{\frac{1}{2}} \tag{3.13}$$

## 3.4 Approximation by B-Spline Curve and Its Curvature Monotonicity

We have shown that when the degree $n$ of the Bézier curve is increased infinitely, its control points converges to points on a log-aesthetic curve. The same control points can be used to define a B-spline curve of degree $m \le n$ and the same discussion leads to that the B-spline converges to the same log-aesthetic curve.

We define a cubic uniform B-spline curve by the control points in this subsection. Especially we will discuss the conditions that guarantee curvature monotonicity when the distance of two consecutive control points is kept constant. As shown in the left of Fig. 2, let four consecutive control points which define a segment of the B-spline curve $C_s(t)$ ($0 \le t \le 1$) be $P_i$, $P_{i+1}$, $P_{i+2}$ and $P_{i+3}$. As curvature monotonicity does not change by scaling, the distance between two control points is normalized to 1 and the positions of the four control points are specified by two angles $\theta_0$ and $\theta_1$. We investigate that for a given $\theta_0$ ($>0$), which value of $\theta_1$ will guarantee curvature monotonicity. Because of the symmetry of the locations of the control points, it is enough to check monotonicity for $\theta_1 < \theta_0$. Let $\kappa$ be the curvature of the curve segment determined by these four control points. Its derivative $d\kappa / dt$ is always negative for any $t \in [0,1]$ as shown in Appendix if the following condition is satisfied:

$$\boldsymbol{sin}\left(\theta_0 + \frac{\theta_1}{2}\right) - 3\boldsymbol{sin}\frac{\theta_1}{2} > 0 \tag{3.14}$$

Hence this condition guarantees curvature monotonicity. If the left-hand side of the above equation is assumed to be equal to $0$, for any $\theta_0$, we can solve it as follows:

$$\theta_1 = 2\boldsymbol{arccos}\left(\frac{3 - \boldsymbol{cos}\,\theta_0}{\sqrt{10 - 6\boldsymbol{cos}\,\theta_0}}\right) \tag{3.15}$$

This $\theta_1$ is the maximum value to guarantee curvature monotonicity.

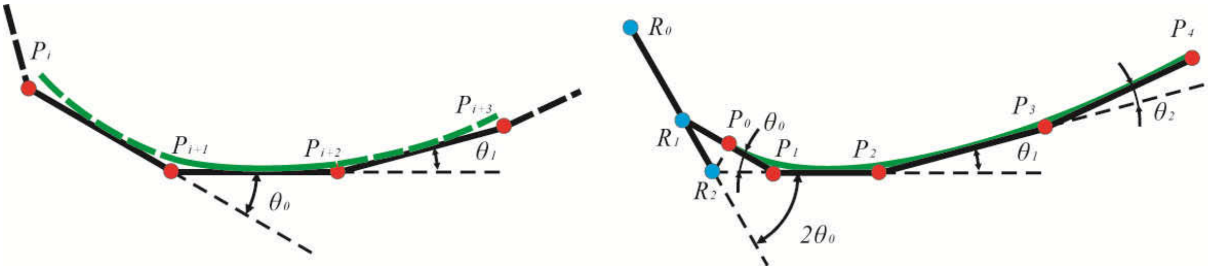


Fig. 2: The location of the control points specified by two angles $\theta_0$ and $\theta_1$.

The important B-spline curve in practice is a curve called the endpoint-interpolating B-spline curve whose start and end points coincide with the first and last control points, respectively. It is generated with a knot vector whose first and end knots are multiple. For a cubic curve, the multiplicity is set to be four. As shown in the right of Fig. 2, the length between the first control point $P_0$ and the second control point $P_1$ is set to be $\boldsymbol{cos}\,\theta_0 / 3$ and the length between $P_1$ and $P_2$ is $2/3$. In the figure, $R_2$ is located at the point where $P_0$ is translated by $1/3$ to the left and $R_1$ is located at a position such that the angle between $P_2R_2$ and $R_2R_1$ is twice large of $\theta_0$. $R_0$ is on $R_2R_1$ and the length of $R_2R_0$ is set to be $1$. The first segment of the endpoint-interpolating B-spline curve is identical to the segment of a uniform cubic B-spline curve whose control points are given by $R_0$, $R_2$, $P_2$ and $P_3$. Similarly the second segment is identical to the segment of a uniform cubic B-spline curve whose control points are given by $R_2$, $P_2$, $P_3$ and $Q_3$. The third segment is not affected by knot multiplicity and it is a segment of a uniform cubic B-spline curve. Therefore the condition expressed by Eq.(3.14) can be directly applied to each segment if we use adequate control points to evaluate the condition. If we guarantee the curvature monotonicity of each segment, we can guarantee curvature monotonicity of the whole curve.

The approximation method of the log-aesthetic plane curve by the non-stationarily typical curve proposed in this paper does not guarantee curvature monotonicity. However we can use Eq.(3.14) to check curvature monotonicity of the whole curve including the endpoint-interpolating B-spline curve and that is very useful in practice.

## 4 APPROXIMATION OF LOG-AESTHETIC SPACE CURVE

In this section, we will propose a method to approximate the log-aesthetic space curve by the non-stationary typical Bézier curve. The space curve has a torsion which is not equal to $0$ and it is necessary to control torsion as well as curvature. Even if a local coordinate system such as the Frenet frame is adopted instead of a global coordinate system, it is not possible to control curvature and torsion independently by changing only the scaling factor with a fixed rotation axis and a fixed rotation angle of the transition matrix since it can adjust the strength of the "bending" of the curve in one direction. Therefore we fix the scaling factor to be 1 and think about adjustments of the rotation axis and the rotation angle. Although the usual Class A Bézier curve is defined in a global coordinate system, our method proposed here uses the Frenet frame, one of local frames. The proposed curve is different from the Class A Bézier curve to two points; one is that the curve is non-stationary typical and the other is to be defined in a local coordinate system.

For a space curve, the slope of the logarithmic curvature graph $\alpha$ is assumed to be not equal to $0$ and also the slope of the logarithmic torsion graph $\beta \neq 0$ to make our discussion simple. It is possible to treat $\alpha = 0$ ® = 0 and/or $\beta = 0$ cases similarly. Then the general equation of the log-aesthetic space curve is given by

$$\rho^{\alpha} = cs + d \tag{4.1}$$

$$\mu^{\beta} = gs + h \tag{4.2}$$

similarly to the plane curve where $\rho$ is the radius of curvature, $\mu$ is the radius of torsion and $s$ is the arc length. $\alpha$, $\beta$, $c$, $d$, $g$ and $h$ are constants and if these values are changed, the shape of the curve will be modified.

For a space curve $C(s)$ given as a function of the arc length $s$, let $t$, $n$ and $b$ be unit tangent, unit normal and unit binormal vectors, respectively. The relationships among these vectors are expressed by the following Frenet-Serret formula:

$$\begin{bmatrix} t'(s) \\ n'(s) \\ b'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & \tau(s) & 0 \end{bmatrix} \begin{bmatrix} t(s) \\ n(s) \\ b(s) \end{bmatrix} \tag{4.3}$$

where $\kappa$ is the curvature and $\tau$ is the torsion. Since $t' = kn$, $\kappa$ caused by the variation of the arc length $s$ represents a rotation ratio of $t$ about $b$. Similarly since $b' = -\tau n$, $\tau$ represents a rotation ratio of $n$ (or $b$) about $t$. Hence the transition matrix is determined by defining a rotation locally by use of the Frenet frame. Similar to that for a plane curve, the amount of the rotation angle (direction angle) $\theta$ of the tangent vector is given by Eqn. (2.13), the amount of the rotation angle $\theta_\kappa$ of $t$ in the $tn$ plane and the amount of the rotation angle $\theta_\tau$ of $n$ (or $b$) in the $nb$ plane are assumed to be given by

$$\theta_\kappa = \frac{\alpha(c_{\kappa 0}s + c_{\kappa 1})^{\frac{\alpha-1}{\alpha}}}{(\alpha-1)c_{\kappa 0}} + c_{\kappa 2} \tag{4.4}$$

$$\theta_\tau = \frac{\beta\left(c_{\tau 0}s + c_{\tau 1}\right)^{\frac{\beta-1}{\beta}}}{(\beta-1)c_{\tau 0}} + c_{\tau 2} \tag{4.5}$$

where $\theta_\kappa$ and $\theta_\tau$ are rotation angles about $b$ and $t$, respectively. Hence if $t$, $n$ and $b$ are considered to be the $x$-, $y$- and $z$-axes, respectively, by using $\Delta\theta_\kappa$ and $\Delta\theta_\tau$ caused by the variation $\Delta s$ of $s$ the unit tangent vector $t$ is rotated by the following rotation matrices defined in the Frenet frame:

$$R_\kappa(\Delta\theta_\kappa) = \begin{bmatrix} \cos\Delta\theta_\kappa & -\sin\Delta\theta_\kappa & 0 \\ \sin\Delta\theta_\kappa & \cos\Delta\theta_\kappa & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.6}$$

$$R_\tau(\Delta\theta_\tau) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Delta\theta_\tau & -\sin\Delta\theta_\tau \\ 0 & \sin\Delta\theta_\tau & \cos\Delta\theta_\tau \end{bmatrix} \tag{4.7}$$

To determine the position of the next control point, the transition matrix should be defined as the combination of these two rotation matrices. These matrices are supposed to be applied alternately for small amounts $\Delta\theta_\kappa$ and $\Delta\theta_\tau$ and it is desirable for the resulting rotation not to be dependent on the order of these rotations. Therefore we use the linear combination of transformations proposed by Alexa [3]. Then the product $A \otimes B$ of matrices $A$ and $B$ is defined as follows:

$$A \otimes B = \lim_{n\to\infty}\left(A^{\frac{1}{n}}B^{\frac{1}{n}}\right)^n \tag{4.8}$$

The left of Fig. 3 shows examples of Bézier curves generated by the method explained so far. The $\alpha$ and $\beta$ values of this curve is $1.5$ and $1.2$, respectively. The curve is rendered as a pipe and its control points are rendered as spheres. To make clear that it is a space curve, the $x$-, $y$- and $z$-axes are drawn as straight lines. The top-right of Fig. 3 shows the curvature and the torsion of curves of degrees $99$, $49$ and $24$ and the bottom right of the same figure shows their logarithmic curvature and torsion graphs. When the degree is $99$, both the LCG and LTG are given by almost straight lines and their slopes are $1.46$ and $1.14$, respectively and they are almost equal to the specified values. The monotonicity of curvature and torsion is kept even if the degree is decreased, but the two logarithmic graphs deviate from straight lines. Especially the third control point does not contribute to torsion because the first three control points determine the osculating plane at the start point. That causes a deviation of the LTG from a straight line. One of our future works is to find a method to keep the linearity of the LTG for a curve of low degree.

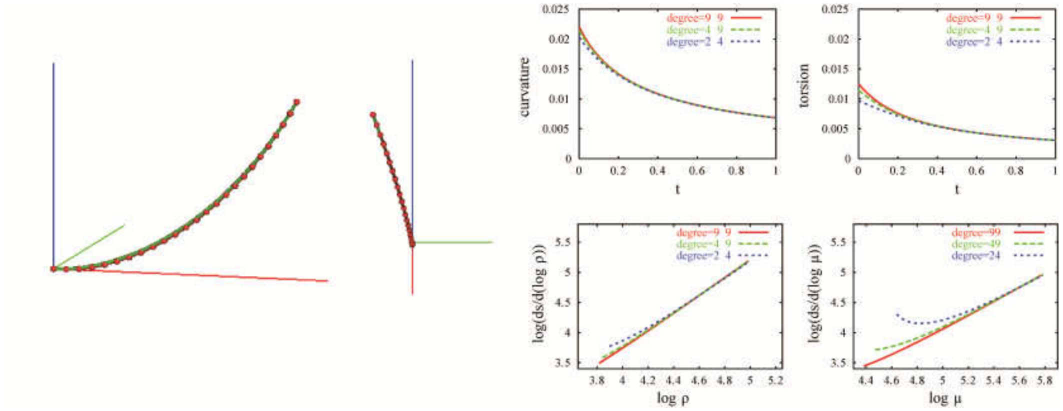


Fig. 3: Examples of approximation of a log-aesthetic space curve by Bezier curves of degree 24, $\alpha = 1.5$, $\beta = 1.2$.

## 5 NON-STATIONARY TYPICAL SURFACE

We extend our methods for the curve to the surface. From the discussions on the space curve in the previous section, the control of the rotation axis and angle with the unit scaling factor is more useful than that of the scaling factor with the fixed rotation axis and angle. Hence for a surface, the distance between two control points in the same parameter direction is fixed and the position of the next control points is determined by a rotation about the diagonal line of each control polygon as shown in the top-left of Fig. 4. In the figure the lengths of two vectors $P_{i,j}P_{i+1,j}$; and $P_{i,j+1}P_{i+1,j+1}$ are kept to be the same and those of $P_{i,j}P_{i,j+1}$ and $P_{i+1,j}P_{i+1,j+1}$ are the same. If the positions of $P_{i,j}$, $P_{i+1,j}$, and $P_{i,j+1}$ are already fixed, the degree of freedom of the position of $P_{i+1,j+1}$ is only the rotation about the diagonal line and $P_{i+1,j+1}$ is rotated and move to $R_{i+1,j+1}$. There is a possibility to generate surfaces of high quality by controlling the rotation angle. Since the definition of the surface is similar to the non-stationary typical curve, we call this type of the surface the non-stationary typical surface.

The top-right of Fig. 4 shows an example of the non-stationary typical surface and its Gaussian and mean curvatures are shown in the bottom left and bottom-right of Fig. 4, respectively. To generate this surface, the distances between two control points in the directions of two parameters u and v are fixed to be constant and the same. The control points $P_{i,0}$ corresponding to the iso-parametric curve of $v = 0$ are generated to approximate a log-aesthetic curve with $\alpha = -0.5$ and similarly the control points $P_{0,j}$ corresponding to the iso-parametric curve of $u = 0$ are generated to approximate a log-aesthetic curve with $\alpha = -0.6$. Furthermore each control polygon is rotated about its diagonal line by $0.01^\circ$. The number of the control points is $25 \times 25$ and by these control points a Bézier patch of bi-25 degree is generated. The surface shown in the section is generated by controlling the rotation angle simply and how to control it and how to generate a surface for given boundary curves should be researched in future.

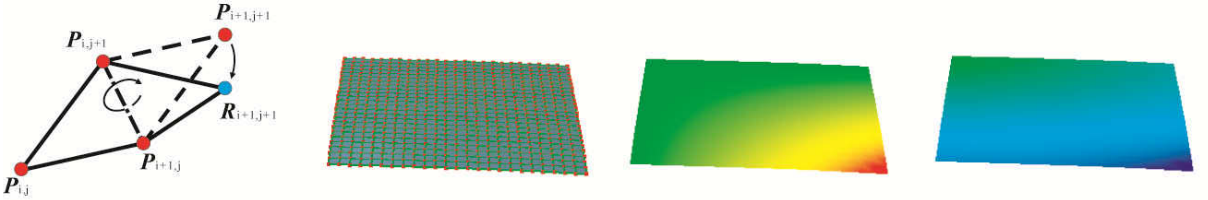


Fig. 4: From left to right, determination of positions of the control points by rotation about the diagonal line of the control polygon, an example of non-stationary typical surface and its Gaussian and mean curvature.

## 6 CONCLUSIONS AND FUTURE WORK

In this paper we have clarified the relationship between the Class A Bézier curve and the slope of the logarithmic curvature graph $\alpha$ which was unclear and proposed methods to approximate the log-aesthetic curves including the space curve by the Bézier and B-spline curves. The transition matrix made as a combination of rotation and scaling is identical for a plane curve defined in either a global coordinate system or a local coordinate system such as the Frenet frame. The typical Bézier curve is included in the class A Bézier curve and the transition matrices of the Class A Bézier curve as well as the typical curve are defined in a global coordinate system. They can be considered to be defined in a local coordinate system. However for a space curve, global and local should be clearly distinguished and we have shown that to use the Frenet frame, one of local coordinate systems is effective to define a curve whose both LCG and LTG are given by almost straight lines. For the surface we have shown only a basic framework and future work on the surface is inevitable.

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**APPENDIX**

**Curvature Monotonicity of Unifrom Cubic B-Spline Curve**

The four control points $Q_0$, $Q_1$, $Q_2$, and $Q_3$ of a cubic Bézier curve $C(t)$ determined by the four control points $P_{i+j}$ ( $j = 0,\cdots,3$) of the B-spline curve in Fig.2 are given by

$$Q_0 = \frac{P_i + 4P_{i+1} + P_{i+2}}{6}, Q_1 = \frac{2P_{i+1} + P_{i+2}}{3} \; Q_2 = \frac{P_{i+1} + 2P_{i+2}}{3} \; Q_3 = \frac{P_{i+1} + 4P_{i+2} + P_{i+3}}{6}$$

In order to simplify resulting expressions, the length of $Q_1Q_2$ is scaled to be 1. Note that curvature monotonicity is not affected by this scaling.

It is sufficient to examine the sign of the derivative of the curvature to detect curvature monotonicity of the curve. Let $g(t)$ be square of the norm of the first derivative vector of the cubic Bézier curve and $f(t)$ be the cross product of this vector and the second derivative vector. Then the curvature of the curve $\kappa(t)$ is given by

$$\kappa(t) = \frac{f(t)}{g(t)^{\frac{3}{2}}}$$

So the derivative $d\kappa(t)/dt$ is given by

$$\frac{d\kappa(t)}{dt} = \frac{\frac{df(t)}{dt}g(t) - \frac{2}{3}f(t)\frac{dg(t)}{dt}}{f(t)^{\frac{5}{2}}}$$

It is sufficient to examine the sign of the numerator of the above equation $h(t)$ to detect curvature monotonicity. Since $f(t)$ is quadratic because of the relationship between the first and second derivatives and $g(t)$ is of degree 4, $h(t)$ is of degree 5. When $h(t)$ is expressed by a Bézier polynomial, we denote its coefficients as $C_i$ ($i = 0,\cdots,5$). Especially $C_5$ is given by

$$C_5 = -\cos^3\frac{\theta_1}{2}\left\{\sin\left(\theta_0+\frac{\theta_1}{2}\right)-3\sin\frac{\theta_1}{2}\right\}$$

As a matter of fact, Eqn. (3.14) is the condition for $C_5$ to be negative. The left of Fig. 5 shows a graph of the left side of Eq.(3.14) for $0 \le \theta_0, \theta_1 \le \pi/2$. This graph is made by rendering the region whose value is positive and plotting flat the region where curvature monotonicity is not guaranteed. The right of Fig.5 renders the region whose value is negative and plotting the other region flat on the contrary to the left graph. Furthermore it renders the negative region of $C_4$ and plots flat the other region. This figure indicates that the two flat regions do not intersect each other and $C_4$ is negative in the region where the value of the left-hand side of Eq.(3.14) is positive. Since similar graphs are plotted for $C_i$ $(i=0,\cdots,3)$, if $C_5 < 0$, $C_i$ $(i=0,\cdots,4)$. Therefore $C_5 < 0$ is a necessary and sufficient condition for $f(t)$ to be always negative.

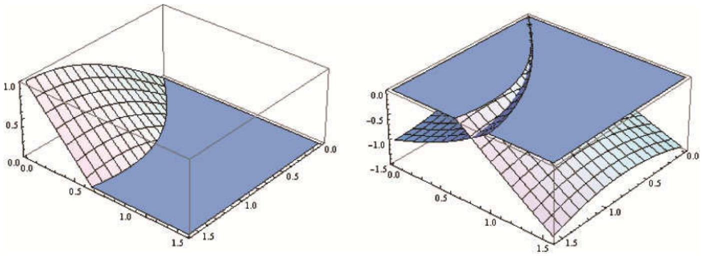


Fig. 4: Comparison of the left-hand side of Eqn. (3.14) and $C_4$