Geometric Interpolation Method in R^3 Space with Optimal Approximation Order

Xiao-Diao Chen[1,2], Weiyin Ma[2] and Jinsheng Zheng[3]

[1]Hangzhou Dianzi University, xiaodiao@nit.zju.edu.cn
[2]City University of Hong Kong, mewma@cityu.edu.hk
[3]Hangzhou Dianzi University, jinsheng957@yahoo.com.cn

ABSTRACT

This paper presents a geometric interpolation method for curve approximation in R^3 space. Given a curve, the new method is to find an approximation Bézier curve of degree 4 tangent with the given curve at the two end points and at an inner point as well. The resulting Bézier curve is explicitly expressed in the parameters of the tangent inner point of both the given curve and the approximation curve. We prove that the approximation order of the new method is 6, which is the optimal approximation order in the traditional conjecture.

Keywords: degree reduction, approximation, inner point interpolation method.
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1 INTRODUCTION

Curve approximation is an important problem in geometric design and it has wide applications in computer aided design and manufacture (CAD/CAM). The existence of the solution, the computation of the solution and the approximation order are three key factors of an approximation scheme. Achieving a good approximation order is an important goal in curve approximation [2,7,9,10,12]. A curve $s \mapsto q(s)$ has an approximation order k to another curve $t \mapsto p(t)$ at a point $p(t_*) = q(s_*)$ if there exists a reparameterization $s = \phi(t)$ such that

$$(\frac{d}{dt})^l (p(t) - q(\phi(t)))|_{t=t_*} = 0, l < k. \tag{1-1}$$

There is a conjecture that smooth curve in R^n can be approximated with an approximation order

$$m = d + 1 + \lfloor (d-1)/(n-1) \rfloor,$$

where $\lfloor r \rfloor = \max\{j \in Z: j \le r\}$, and d is the degree of the approximation curve [9,10,12].

For planar curve approximation, an accurate cubic interpolation scheme with an approximation order of 6 for planar curves was first constructed by de Boor, Hollig and Sabin [1]. The method used positions, derivatives and curvatures at both of the two end-points for determining the resulting approximation curve. Chen et al. [2] present an inner point interpolation (IPI) method for constructing a cubic Bézier curve that is tangent with the given curve at three positions, i.e., the two end points and an inner point. They prove that the optimal approximation order is also 6 for the cubic case. Both the two methods in [1,2] are not sensitive to the parameterization formula of the given curve. Numerical examples in [2] also show that both the standard Hermite method and G^2 Hermite method that matches only directions of derivatives up to the second order are sensitive to the parameterization and their approximation order tends to be 4. Several other approximation schemes with optimal orders have been proposed for approximating curves with low degrees [3,4,6,8-13]. Scherer claimed for the proof of the conjecture in one of his unpublished manuscripts in 1994 [9].

For approximation curves of degree $n \geq 4$, there are few references discussing approximation schemes with the optimal order. Several conjectures are made on the optimal order in R^d by Höllig, Koch, and Rababah [8-10,12]. The corresponding optimal orders are conjectured by analyzing the relation between the number of the constrained equations of the derivatives on the two end points and the number of the unknown variables. Note that the r-th order derivatives and their directional vectors are sensitive to the parameterization of a curve and the geometric Hermite method is therefore also sensitive to the parameterization.

This paper discusses the curve approximation problem in R^3 space by using quartic Bézier curves. It provides a geometric interpolation method, which is extended from the IPI method in [2]. For approximating a given curve, the new method is to find a Bézier curve of degree 4, which is tangent with the given curve at the two end-points and an inner tangent point. The resulting curve is explicitly expressed in the parameters of the tangent inner point of both the given curve and the approximation curve. We prove that the optimal approximation order of the new method is also 6, which is the optimal order in the traditional conjecture in [9,10,12]. Numerical examples show both the optimal approximation order and the approximation effect of the new method. Analyzing the additional degrees of freedom leads to the possibility of obtaining an approximation curve which interpolates an additional inner point in addition to the interpolation with respective tangent direction at the two end points and one inner point. However, a detailed discussion and the corresponding proof on whether such an approximation curve exists are beyond the scope of this paper.

The outline of this paper is as follows. In section 2, we explain the geometric interpolation method in R^3 space. In section 3, we prove that the optimal approximation order of the new method is also 6. We also point out that it is possible to obtain an approximation order of 7 by analyzing the degree of freedom of the equations and unknown variables as well. In section 4, we provide several numerical examples.

2 GEOMETRIC INTERPOLATION METHOD IN R^3 SPACE

Suppose that the given curve and the quartic approximation Bézier curve are represented by

$$\mathbf{C}(t) = \begin{pmatrix} X_C(t) \\ Y_C(t) \\ Z_C(t) \end{pmatrix}, t \in [0,1], \tag{2-1}$$

and

$$\mathbf{A}(u) = \sum_{i=0}^{4} \mathbf{Q}_i B_i^4(u) = \begin{pmatrix} X_A(u) \\ Y_A(u) \\ Z_A(u) \end{pmatrix}, u \in [0,1], \tag{2-2}$$

respectively, where $B_i^4(u) = \binom{4}{i}(1-u)^{4-i}u^i$, for i=0, 1, …, 4, are the Bernstein basis functions and $\{\mathbf{Q}_i\}$ are the unknown control points. From the constraint that $\mathbf{C}(t)$ and $\mathbf{A}(u)$ are tangent with each other at the two end points, we have

$$\begin{aligned} &\mathbf{Q}_0 = \mathbf{C}(0), \mathbf{Q}_1 = \mathbf{Q}_0 + \alpha \mathbf{C}'(0), \\ &\mathbf{Q}_4 = \mathbf{C}(1), \mathbf{Q}_3 = \mathbf{Q}_4 - \beta \mathbf{C}'(1), \end{aligned} \tag{2-3}$$

To completely define the approximation curve, one should determine both the control point $\mathbf{Q}_2(x_2, y_2, z_2)$ and the values of α and β. Altogether, there are five unknown variables. When we constrain $\mathbf{C}(t)$ and $\mathbf{A}(u)$ to be tangent to each other at the inner point of $\mathbf{C}(t_1)$, we have

$$\begin{cases} X_A(u_1)-X_C(t_1)=0, \\ Y_A(u_1)-Y_C(t_1)=0, \\ Z_A(u_1)-Z_C(t_1)=0, \\ X_A'(u_1)Y_C'(t_1)-Y_A'(u_1)X_C'(t_1)=0, \\ X_A'(u_1)Z_C'(t_1)-Z_A'(u_1)X_C'(t_1)=0. \end{cases} \tag{2-4}$$

The five equations in the equation system (2-4) are all linear in the five unknown variables x_2, y_2, z_2, α and β. By directly solving the above linear equation system in x_2, y_2, z_2, α and β, we obtain

$$\begin{cases} x_2 = x_2(u_1,t_1), \\ y_2 = y_2(u_1,t_1), \\ z_2 = z_2(u_1,t_1), \\ \alpha = \alpha(u_1,t_1), \\ \beta = \beta(u_1,t_1). \end{cases} \tag{2-5}$$

Remark 1: The details of the derivation of Eqs. (2-4) and (2-5) are given in Appendix 1.

Once the values of t_1 and u_1 are given, the values of x_2, y_2, z_2, α and β are directly computed from Eq. (2-5), and thus the resulting quartic Bézier curve is obtained.

With the help of the software Maple, it can be verified that

$$\alpha(u_1,t_1)=\frac{h_4(u_1,t_1)}{h_3(u_1,t_1)u_1},\quad \beta(u_1,t_1)=\frac{g_4(u_1,t_1)}{(u_1-1)u_1^3},$$

where $h_4(u_1,t_1)$, $h_3(u_1,t_1)$ and $g_4(u_1,t_1)$ are polynomials in u_1 of degree 4, 3 and 4, respectively. We will firstly select a suitable value of t_1 such that there exists an interval $[u_s,u_e]$ satisfying $\alpha\,(u,t_1) > 0$, $\beta\,(u,t_1) > 0$, for $u\in(u_s,u_e)$, where $u_s,u_e\in(0,1)$. When t_1 has been set, u_s and u_e are one of the real roots of the polynomial equations $h_4(u,t_1)$ and $g_4(u,t_1)$, respectively, which can be directly computed through the corresponding explicit formulae. Finally, the value of u_1 can be simply set as $(u_s+u_e)/2$.

Note that, instead of setting the two free unknown parameters u_1 and t_1 in the equation system (2-5) as mentioned above, these two free parameters can also be further exploited to meet other additional interpolation constraints. It should be possible for adding one more interpolation constraint such that the approximation curve also interpolate a second inner point of the given curve $\mathbf{C}(t_2)$, and we obtain

$$\begin{cases} X_A(u_2)-X_C(t_2)=0, \\ Y_A(u_2)-Y_C(t_2)=0, \\ Z_A(u_2)-Z_C(t_2)=0. \end{cases} \tag{2-6}$$

By substituting Eq. (2-5) into the equation system (2-6), we have three equations

$$\begin{cases} H_1(u_1,u_2,t_1,t_2)=0 \\ H_2(u_1,u_2,t_1,t_2)=0 \\ H_3(u_1,u_2,t_1,t_2)=0 \end{cases} \tag{2-7}$$

in four unknown variables u_1, u_2, t_1 and t_2. If the equation system (2-7) has a valid solution, we will show at the end of the next section that the approximation order can be further increased to 7, which is larger than that of the traditional conjecture. However, equation system (2-7) is quite complicated and we have not proven whether it has a valid solution in the general case.

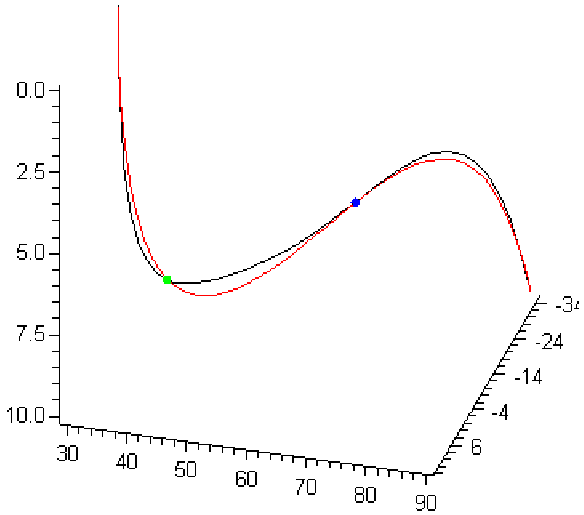

Fig. 1: The existence of a solution of the equation system (2-7). The solid curve in black and the dashed curve in red are the given curve and the approximation curve, while the points in blue and in green are the tangent point and another interpolation point, respectively.

Fig. 1 shows such a numerical example, which has a valid solution for the equation system (2-7). In Fig. 1, the given curve is a Bézier curve of degree 7, which has the control points (30,-10,0), (40,20,5), (45,20,10), (55,15,0), (65,0,10), (70,-10,5), (80,-30,0) and (90,-35,10). The quartic approximation curve has the control points (30,-10,0), (47.1606,41.4817,8.5803), (61.4018,3.7096,8.7329), (78.6527,-29.3263,-1.3473) and (90,-35,10), which is tangent with the given curve at point (69.6205, -9.6420, 5.0391) and interpolates it at point (46.3603, 13.6615, 5.4489) at the same time. As mentioned in the previous section, however, we cannot prove at the moment whether it has a valid solution or not for general cases since the equation system (2-7) is quite complicated.

3 THE OPTIMAL APPROXIMATION ORDER

From the formula (2-5), there always exists an approximation curve **A**(u) which is tangent with the given curve **C**(t) at three points, i.e., the two end points and an inner point. Each tangent point is a double intersection point, whose multiplicity is 2. So the two curves **C**(t) and **A**(u) will interpolate each other six points counting multiplicities. We will show the corresponding approximation order is 6. Suppose that the tangent points are **A**(0)=**C**(0), **A**($\hat{u}_1$) = **C**($\hat{t}_1$), and **A**(1) = **C**(1). There exist three real numbers $\{\gamma_i\}_{i=0}^{2}$ such that **A'**(0) = γ_0**C'**(0), **A'**($\hat{u}_1$) = γ_1**C'**($\hat{t}_1$), and **A'**(1) = γ_2**C'**(1).

Theorem 1: The approximation curve **A**(u) has an approximation order of 6 to the given curve **C**(t).

Proof. From the definition, it needs to prove that there exists a point where the approximation curve **A**(u) has an approximation order of 6 to the given curve **C**(t). Let $\phi(t)$ be a polynomial such that

$$\phi(0) = 0,\ \phi(1) = 1,\ \phi(\hat{t}_1) = \hat{u}_1,$$
$$\phi'(0) = \gamma_0,\ \phi'(1) = \gamma_2,\ \phi'(\hat{t}_1) = \gamma_1.$$

Let $\hat{\mathbf{A}}(t) = \mathbf{A}(\phi(t))$. It can be verified that

$$\hat{\mathbf{A}}(0) = \mathbf{C}(0),\ \hat{\mathbf{A}}(1) = \mathbf{C}(1),\ \hat{\mathbf{A}}(\hat{t}_1) = \mathbf{C}(\hat{t}_1),$$

$$\hat{\mathbf{A}}'(0) = \mathbf{C}'(0),\ \hat{\mathbf{A}}'(1) = \mathbf{C}'(1),\ \hat{\mathbf{A}}'(\hat{t}_1) = \mathbf{C}'(\hat{t}_1).$$

Let $\mathbf{H}(t) = \hat{\mathbf{A}}(t) - \mathbf{C}(t)$. Then we have

$$\begin{cases} \mathbf{H}(0) = \mathbf{0}, \mathbf{H}(1) = \mathbf{0}, \mathbf{H}(\hat{t}) = \mathbf{0}, \\ \mathbf{H}'(0) = \mathbf{0}, \mathbf{H}'(1) = \mathbf{0}, \mathbf{H}'(\hat{t}) = \mathbf{0}. \end{cases}$$

Note that there is a unique polynomial $\hat{\mathbf{H}}(t) \equiv 0$ of degree 5 such that

$$\begin{cases} \hat{\mathbf{H}}(0) = \mathbf{0}, \hat{\mathbf{H}}(1) = \mathbf{0}, \hat{\mathbf{H}}(\hat{t}) = \mathbf{0}, \\ \hat{\mathbf{H}}'(0) = \mathbf{0}, \hat{\mathbf{H}}'(1) = \mathbf{0}, \hat{\mathbf{H}}'(\hat{t}) = \mathbf{0}, \end{cases}$$

we have then

$$\mathbf{H}^i(0) = \mathbf{0}, i = 0,1,\cdots,5, \tag{3-1}$$

where $\mathbf{H}^i(0)$ is the value of the i-th derivative function of $\mathbf{H}(t)$ at t=0. Note that $\mathbf{H}^i(0) = \hat{\mathbf{A}}^i(0) - \mathbf{C}^i(0)$, combining with Eq.(3-1), we have

$$\left(\frac{d}{dt}\right)^l (\mathbf{C}(t) - \mathbf{A}(\phi(t)))|_{t=0} = 0, l < 6.$$

From the definition of Eq. (1-1), the approximation curve $\mathbf{A}(u)$ has an approximation order of 6 to the given curve $\mathbf{C}(t)$. □

Remark 2: A similar proof of Theorem 1 has been given in [2], which is under review, for planar cubic curve approximation. Here, we give the complete proof for easy reference.

Remark 3: If there always exists an approximation curve which is tangent with the given curve and also interpolates one more inner point at the same time, we can similarly prove that the corresponding optimal approximation order is 7.

4 NUMERICAL EXAMPLES

This section provides some numerical examples to illustrate both the effect and the approximation order of the new method.

Example 1: In Fig. 2, the given curve is a helix curve determined by

$$\mathbf{C}(t) = (\cos t, \sin t, t),\ t \in [0, 1],$$

which is also used in [9]. Table 1 shows the approximation results. In Table 1, j is an index for mapping to the parameter interval $[0, 2^{1-j}]$, e_j denotes the corresponding approximation error, and

$m_j = \log_2 \dfrac{e_j}{e_{j+1}}$, which illustrates the approximation order. As shown in Table 1, the approximation order of the new method is 6.

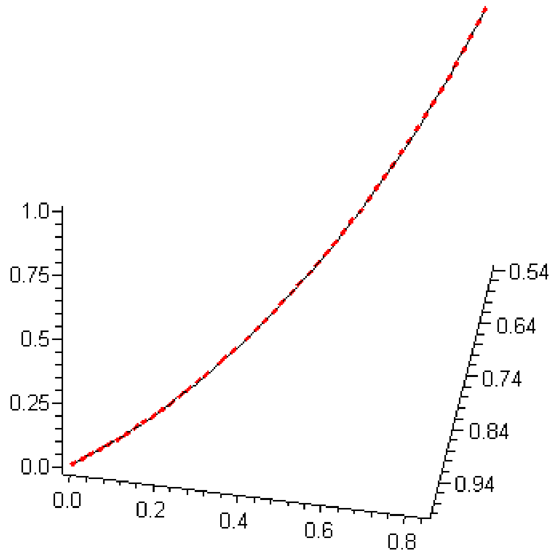

Fig. 2: The solid curve in black is the given helix curve, while the dotted curve in red is the approximation curve.

j	e_i	m_i
1	7.6758e-06	
2	1.1966e-07	6.0032
3	1.8686e-09	6.0008
4	2.9193e-11	6.0002
5	4.5612e-13	6.0000
6	7.1268e-15	6.0000
7	1.1135e-16	6.0000
8	1.7399e-18	6.0000
9	2.7186e-20	6.0000
10	4.2479e-22	6.0000

Tab. 1: Approximation results of a helix curve.

Example 2: In Fig. 3, the given curve is a Bézier curve of degree 7 with control points (30,-10,0), (40,20,5), (45,20,10), (55,15,0), (65,0,10), (70,-10,5), (80,-30,0) and (90,-35,10). Table 2 shows the approximation results. In Table 2, j is again an index for mapping to the parameter interval $[0, 2^{1-j}]$, e_j denotes the corresponding approximation error, and $m_j = \log_2 \frac{e_j}{e_{j+1}}$, which illustrates the approximation order. Table 2 shows again that the approximation order of the new method is 6.

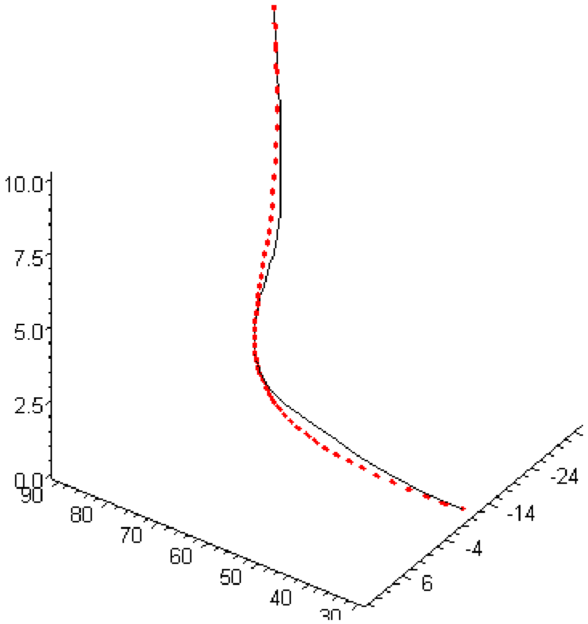

Fig. 3: The solid curve in black is the given Bézier curve of degree 7, while the dotted curve in red is the approximation curve.

j	e_i	m_i
1	1.6856e-00	
2	1.3706e-02	6.9422
3	9.6521e-05	7.1498
4	1.2345e-06	6.2887
5	1.8035e-08	6.0970
6	2.7426e-10	6.0391
7	4.2337e-12	6.0174
8	6.5775e-14	6.0082
9	1.0248e-15	6.0040
10	1.5992e-17	6.0019

Tab. 2: Approximation results of a helix curve.

A few more examples are shown in Fig. 4. Fig. 4(a) shows a helix curve. For this example, the entire curve is approximated with 64 segments for meeting the required tolerance of 10^{-4}. Fig. 4 (b) shows a curve on a surface. This curve is approximated with 16 segments for meeting the required tolerance of 10^{-8}. Fig. 4 (c) and Fig. 4 (d) show curves on a cup and a teapot, respectively. These curves are approximated with 8 segments for meeting the required tolerance of 10^{-4}.

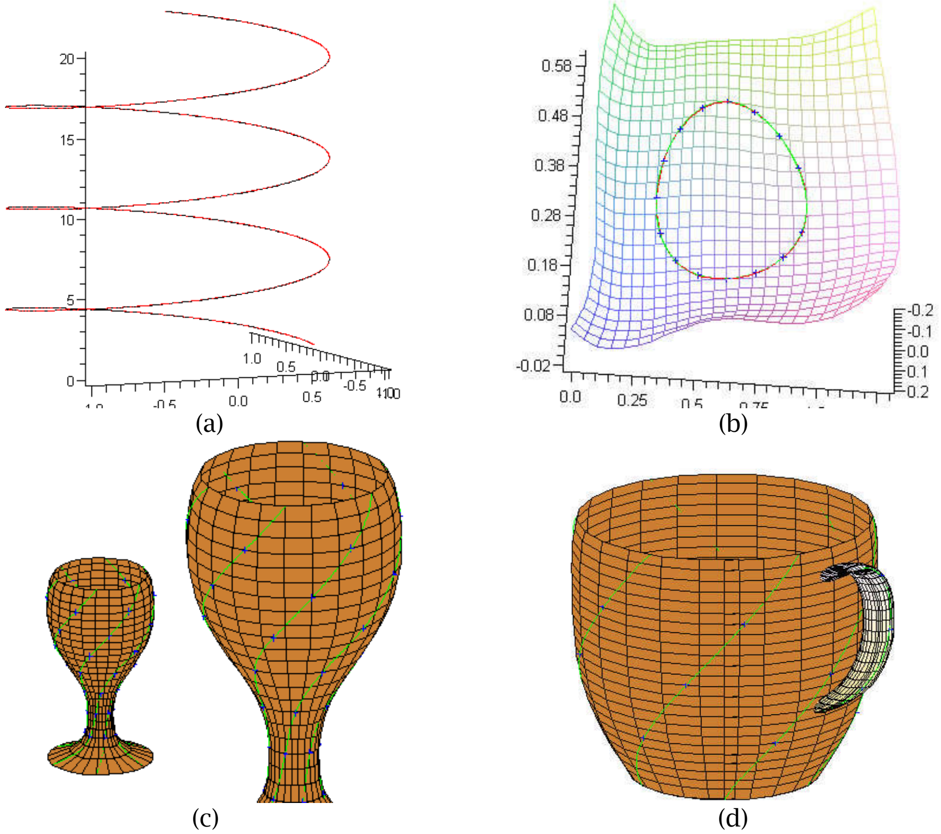

Fig. 4: (a) Approximating a helix curve with 64 segments for a tolerance of 10^{-4}; (b) Approximating a curve on a surface with 16 segments for a tolerance of 10^{-8}; (c) Approximating curves on a cup shape surface; and (d) Approximating curves on a teapot.

5 CONCLUSIONS

This paper presents a geometric interpolation method in R^3 space, which is extended from the IPI method of [2] for planar curve approximation. In the new method, the resulting quartic Bézier curve is explicitly expressed in two variables u_1 and t_1, i.e., the parameters of the tangent point on both the given curve and the approximation curve, which are free for optimization. A method is provided for presetting the values of u_1 and t_1. The new method reaches an approximation order of 6, which is the optimal order in the traditional conjecture. Some examples are presented to show the approximation order and the approximation effect.

By analyzing the degree of freedom of the resulting equations and the corresponding unknown variables, the geometric interpolation method discussed in this paper can be likely extended to a new method with a higher approximation order. Instead of passing through the given curve at the two end points and an interior point with respective tangent directions, the approximation curve can be forced to pass through an additional interior point by exploring the extra degree freedom of the parametrization. As a result, it is possible to achieve an optimal approximation order of 7, which is higher than that of the traditional conjecture in [8,9,12]. The idea was also verified with some preliminary examples. The extension and the proof of the existence of a quartic Bézier curve with an approximation order of 7 will be further addressed in our future work. The optimal selection of the values of u_1 and t_1 can also be discussed for achieving the best approximation effect.

In principle, the geometric interpolation method can also be further extended to the approximation problem by using Bézier curves of higher degrees. While this paper uses a quartic Bézier curve to illustrate the geometric interpolation method, the idea can also be applied to other curves in the future.

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REFERENCES

[1] de Boor, C.; Höllig, K.; Sabin, M.: High accuracy geometric Hermite interpolation, Computer Aided Geometric Design, 4(4), 1987, 269–278.

[2] Chen, X.; Ma, W.; Yong, J.; Paul, J.: Inner point interpolation method with tangent direction constraint for planar curve approximation, Submitted to Computer Aided Geometric Design, 2009.

[3] Degen, W.: Best approximations of parametric curves by splines, In: T. Lyche and L.L. Schumckaker, eds., Mathematical Methods in Computer Aided Geometric Design II, 1992, 171–184.

[4] Degen, W.: High accurate rational approximation of parametric curves, Computer Aided Geometric Design, 10(3-4), 1993, 293–313.

[5] Elber, G.; Lee, I.; Kim, M.: Comparing offset curve approximation methods, IEEE Computer Graphics and Applications, 17 (1), 1997, 62–71.

[6] Farin, G.: Geometric Hermite interpolation with circular precision, Computer-Aided Design, 40(4), 2008, 476–479.

[7] Gasper, J.; Jernej, K.; Marjeta, K.; Emil, Z.: On geometric interpolation by planar parametric polynomial curves, Mathematics of Computation, 76 (260), 2007, 1981–1993.

[8] Hö11ig, K.: Algorithms for rational spline curves, Transactions of the Fifth Army Conference on Applied Mathematics and Computing, 1988, 287–300.

[9] Höllig, K.; Koch, J.: Geometric Hermite interpolation, Computer Aided Geometric Design, 12(6), 1995, 567–580.

[10] Höllig, K.; Koch, J.: Geometric Hermite interpolation with maximal order and smoothness, Computer Aided Geometric Design, 13 (8), 1996, 681–695.

[11] Marjeta, K.: Geometric Hermite interpolation by cubic G^1 splines, Nonlinear Analysis, 70(7), 2009, 2614–2626.

[12] Rababah, A.: High accuracy Hermite approximation for space curves in R^d, Journal of Mathematical Analysis and Applications, 325(2), 2007, 920–931.

[13] Schaback, R.: Interpolation with piecewise quadratic visually C^2 Bézier polynomials, Computer Aided Geometric Design, 6(3), 1989, 219–233.

Appendix 1: Derivation of Eqs. (2-4) and (2-6).

In principle, Eqs. (2-4) and (2-5) can be derived for general cases. For the sake of simplicity, with a suitable affine transformation, we assume that the two end points of the given curve are given as **C**(0)=(0,0,0) and **C**(1) =(1,0,0), their directional vectors of the derivatives at the two end points are (1,1,1) and $(1,-1,z_e)$. Without loss of generality, let the inner point and the directional vector of its derivative be (x_t, y_t, z_t) and $(1,0,z_m)$. The approximation curve then becomes

$$\mathbf{A}(u)=\begin{pmatrix} 4\alpha(1-u)^3u+6x_2(1-u)^2u^2+4(1-\beta)(1-u)u^3+u^4 \\ 4\alpha(1-u)^3u+6y_2(1-u)^2u^2+4\beta(1-u)u^3 \\ 4\alpha(1-u)^3u+6z_2(1-u)^2u^2-4\beta z_e(1-u)u^3 \end{pmatrix}.$$

Substituting **A**(u) into Eq. (2-4), we obtain

$$\begin{cases} 4\alpha(1-u_1)^3u_1+6x_2(1-u_1)^2u_1{}^2+4(1-\beta)(1-u_1)u_1{}^3+u_1{}^4-x_t=0, \\ 4\alpha(1-u_1)^3u_1+6y_2(1-u_1)^2u_1{}^2+4\beta(1-u_1)u_1{}^3-y_t=0, \\ 4\alpha(1-u_1)^3u_1+6z_2(1-u_1)^2u_1{}^2-4\beta z_e(1-u_1)u_1{}^3-z_t=0, \\ 12\alpha(1-u_1)^2u_1-4\alpha(1-u_1)^3+12y_2(1-u_1)u_1{}^2 \\ \qquad -12y_2(1-u_1)^2u_1+4\beta u_1{}^3-12\beta(1-u_1)u_1{}^2=0, \\ (-12\alpha(1-u_1)^2u_1+4\alpha(1-u_1)^3-12x_2(1-u_1)u_1{}^2 \\ \qquad +12x_2(1-u_1)^2u_1-4(1-\beta)u_1{}^3 \\ \qquad +12(1-\beta)(1-u_1)u_1{}^2+4u_1{}^3)z_m \\ \qquad +12\alpha(1-u_1)^2u_1-4\alpha(1-u_1)^3+12z_2(1-u_1)u_1{}^2 \\ \qquad -12z_2(1-u_1)^2u_1-4\beta z_eu_1{}^3+12\beta z_e(1-u_1)u_1{}^2=0. \end{cases}.$$

Finally, Eq. (2-5) becomes

$$\begin{cases} \alpha=\dfrac{z_m(x_t+2u_1{}^3+y_t-2u_1x_t+2u_1-2y_tu_1-u_1{}^4)+2z_e\cdot y_tu_1-z_ey_t-z_t}{2u_1(u_1-1)^3(-2z_m+1+z_e)}, \\ \beta=-\dfrac{z_m(u_1{}^4-2u_1{}^3+y_t-2y_tu_1+2u_1x_t-x_t)+2y_tu_1-2u_1z_t+z_t-y_t}{2(u_1-1)u_1{}^3(1+z_e-2z_m)}, \\ x_2=\dfrac{4y_tu_1-2y_t-4u_1{}^3+3u_1{}^4+x_t}{6u_1{}^2(1-u_1)^2}, \\ y_2=\dfrac{z_m(8u_1{}^3-4u_1{}^4-8u_1x_t+4x_t-2y_t)+(8u_1-4)z_t+(4z_eu_1-4u_1-z_e+3)y_t}{6(u_1-1)^2(1+z_e-2z_m)u_1{}^2}. \\ z_2=\dfrac{z_ez_m(2u_1{}^4-4u_1{}^3+(4u_1-2)(x_t-y_t))}{6(1-u_1)^2u_1{}^2(1+z_e-2z_m)} \\ \qquad +\dfrac{+2z_m(2u_1{}^3-u_1{}^4-2y_tu_1-z_t-2u_1x_t+x_t+y_t)}{6(1-u_1)^2u_1{}^2(1+z_e-2z_m)} \\ \qquad +\dfrac{4\cdot u_1z_t-4z_eu_1z_t+8z_ey_tu_1-z_t+3z_ez_t-4z_ey_t}{6(1-u_1)^2u_1{}^2(1+z_e-2z_m)}. \end{cases}$$