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Adaptive Meshing of 2D Heterogeneous Objects Using Material Quadtree

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ABSTRACT

In finite element analysis (FEA), adaptive meshing of an object is usually preferred. With adaptive meshing, geometric accuracies of the mesh and more accurate FEA solution can be achieved while maintaining the computational efficiency. Numerous researches about adaptive meshing have been done and most of the existing schemes can generate meshes properly when the objects under meshing are homogeneously constituted. With the advent of heterogeneous objects, traditional adaptive meshing schemes become inadequate as the material heterogeneities of the mesh nodes and mesh elements are not taken into consideration. Inaccurate FEA results may result when a traditional adaptive mesh for a heterogeneous object is used as input geometry for the FEA. To cope with this problem, proper adaptive meshing schemes for heterogeneous objects should be developed. In this paper, the problem in 2D case is considered and a material quadtree is introduced. In this material quadtree, apart from the geometries of a heterogeneous object, the material heterogeneity information is also taken into account. Different types of quadrants are defined and a material composition variation threshold is introduced. This threshold and the traditional quadtree resolution are used to verify whether a quadrant should be recursively subdivided. By checking these threshold and resolution properly, the resultant material quadtree is able to represent the geometries of the heterogeneous object to predefined level of details. Moreover the material composition variation inside each quadrant should be within a preset threshold. By triangulating the quadrants of the material quadtree, an adaptive mesh can be created.

Keywords: adaptive meshing, heterogeneous object, material quadtree, quadrant.
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1 INTRODUCTION

Heterogeneous objects are objects made of different constituent materials or spatially different structures [11], [19]. In these objects, different materials can be mixed in appropriate proportions so that some desired material properties that are unattainable in homogeneous objects can possibly be designed. Different or even conflicting design requirements can thus be satisfied by these heterogeneous objects. In most cases, heterogeneous objects possess superior performances over homogeneous ones in terms of mechanical, thermal or electrical properties. As a result, heterogeneous objects have attracted considerable research interests in CAD, CAM, CAE and RP fields over the past decade. To produce a heterogeneous object, the object must first be modeled. Generally, two fundamental processes are performed in modeling heterogeneous objects – geometric modeling and material modeling. The former focuses on representing the geometric and topologic information of the objects while the latter targets on modeling material distributions defined over the geometric domain. Some object construction tools (e.g. extrusion, lofting, patterns and Boolean operations etc.) have been developed for manipulating heterogeneous objects [12], [21], [25], [27]. Numerous heterogeneous object representation schemes that can handle the geometry and material of a heterogeneous object have been introduced and some of them can be found in [1], [10], [12], [14-15], [25]. Using these schemes, the information of a heterogeneous object can be output for other downstream applications, such as FEA.

Many FEA based approaches have been proposed for function analysis or design validations of heterogeneous objects [2], [5], [9], [28]. Normally, the material heterogeneity of an object is taken into account and different materials are allowed to be defined for each node or element of the object's mesh in these methods. However, classic mesh generation schemes are usually employed. In this case, only the geometric accuracies of the mesh is guaranteed while the material heterogeneities of the object are poorly characterized in the mesh, as the material information of the object is not taken into consideration during these classic mesh generation processes. To improve material accuracy in interior regions of the object, meshes with higher densities would be used. The drawback in doing this is that the computational efficiencies would be significantly degraded [32].

Thus, proper adaptive meshing algorithms that also take the material information of a heterogeneous object into account should be developed. In this paper, the adaptive meshing problem for 2D heterogeneous objects is investigated. The quadtree concept that is frequently applied in 2D mesh generation process will be modified and a material quadtree is introduced. In traditional quadtree based representation, a 2D space that encloses the object is recursively subdivided into four quadrants until a preset quadtree resolution is reached. Normally, smaller quadrants are used to represent the boundary of the object while larger quadrants are used to represent the interior of the object. In the proposed material quadtree, smaller quadrants might also be subdivided in the interior regions where local material compositions exhibited large variations. An adaptive mesh in both geometry and material aspects can then be created based on the material quadtree of the heterogeneous object. In the following sections, the details of the material quadtree and the process for generating an adaptive mesh from the material quadtree will be discussed.

2 REVIEWS

There are numerous finite element mesh generation schemes and they can be roughly classified into four categories: the mapping method, the node connection method, the grid code based method and the advancing front method. In the mapping method, meshes are first generated in parametric space and the meshes are subsequently mapped to Cartesian coordinate system using diverse mapping techniques [3], [7], [33]. To adaptively mesh an object, the size and shape of the mesh elements must be controlled in the parametric domain. Normally, surface curvature is used as the metric, i.e. large mesh elements are generated in planar or quasi-planar regions while fine-tuned mesh elements are used to facet curved surfaces. The node connection methods consist of two steps. A node grid is first generated for discretizing the object geometries. Some nodes connection techniques are then applied to generate the meshes, for example the Delaunay triangulation method [17]. In grid code based approaches, regular grids/elements superimposing the entire object are used. Normally, the elements straddling the boundaries are further split to make the meshes closely conform to the boundary shapes. Some examples of this approach are the modified quadtree methods [29-30] and the combined octree/Delaunay methods [23]. In the advancing front methods, mesh elements are first generated on the boundaries of the domain and additional elements are consecutively added to the interior until the entire domain is fully covered [16], [18].

Some of the mesh generation schemes are suitable for adaptive mesh generation, for instance, the Delaunay triangulation method, advancing front method, mesh generation using contours, quadtree/octree techniques and mesh refinement by subdivision. In these methods, the geometric compatibility and the topological compatibility of the finite element meshes are taken into account so that the final mesh could be closely conformable to the object shapes and all the mesh elements are properly connected with correct adjacency relationship.

In the case of meshing heterogeneous objects, the physical compatibility should also be considered. In [26], an adaptive meshing method for multi-material objects was proposed. However, multi-material objects are limited in material heterogeneity. Schimpf et al. [22] studied the adaptive meshing problem for human organs (e.g. heart, liver), each of which is also regarded as components with “distinct” materials. Shin [24] proposed an adaptive meshing method for functionally graded material objects. In his method, iso-material contours of the object are created so that the continuous material gradation is converted into step-wise variation. Triangular meshes are then generated inside each iso-material region (which is considered as a homogeneous region) formed from the contours. However, this method can only be applied in two-dimensional cases and the material gradation must be unidirectional. In [20], a quadtree-based mesh generation method was proposed. In this method, heterogeneous spatial data in the format of image (e.g. X-ray, computer assisted tomography, magnetic resonance imaging, etc.) forms the input. The application of this method may possibly be extended to 2D heterogeneous objects. In this case, a heterogeneous object must first be discretized into a number of equal-sized pixels and it is assumed that there is no variation on the material composition inside a pixel. However, this process would be computationally expensive and appropriate algorithms are needed for discretizing the object.

3 APADTIVE MESHING PROCESS

In this proposed adaptive meshing process, a material quadtree is first created to represent a 2D heterogeneous object. This material quadtree is then balanced and the quadrants are triangulated to form a mesh. To improve the quality of the mesh, mesh modification and optimization operations should be carried out. In Figure 1, the proposed workflow for generating a mesh from a 2D heterogeneous object is shown. In the below explanation, a 2D heterogeneous object is represented by a Heterogeneous Feature Tree (HFT). In this representation, an object can be a non-manifold one. The boundary of a heterogeneous object is termed as material boundary. An example HFT is shown in Figure 2. The details of HFT can be found in [12-13].

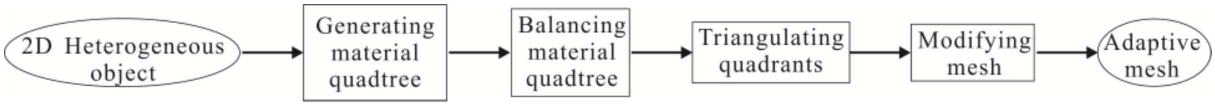


Fig. 1: The proposed workflow for generating an adaptive mesh for a 2D heterogeneous object.

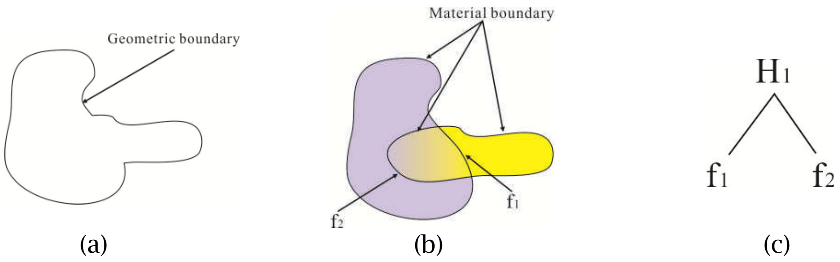


Fig. 2: (a) A 2D geometric model. (b) A heterogeneous model H_1. The boundaries of the object are called material boundaries. (c) The HFT of H_1.

3.1 Terms of Material Quadtree

3.1.1 *Material Composition*

The material composition of any points inside or on a heterogeneous object can be calculated based on the HFT representation of the object. In HFT, a heterogeneous object consists of n number of materials and the material composition M_i of a point i is represented as

$$M_i = (P_{mi1}, p_{mi2}, \ldots, p_{\min}) \quad \text{s.t.} \quad \sum_{k=1}^{n} p_{mik} = 1 \tag{1}$$

where p_{mik} is the composition of material k (k = 1, …, n).

The material composition difference d_{ij} between two points i and j of a heterogeneous object is the magnitude of the difference of their material compositions, i.e.

$$d_{ij} = \left\| M_i - M_j \right\| \tag{2}$$

3.1.2 *Material Composition Variation Threshold and Quadtree Resolution*

In the material quadtree, a material composition variation threshold α is introduced. During the material quadtree generation process, m points in a quadrant are sampled and their material compositions are calculated. The material composition differences between these sampled points are then calculated by equation (2) and the maximum value is obtained, i.e.

$$d_{\max} = Max(d_{ij}) \quad \forall \text{ i, j where i} \neq \text{j and i, j} = 1, \ldots, \text{m} \tag{3}$$

This maximum value is considered as the maximum material composition variation of the quadrant. By comparing $d_{\max}$ with the material composition variation threshold, the quadrant would be determined whether it is subdivided, i,e,

$$\text{subdivide if } d_{\max} > \alpha \tag{4}$$

As a result, the maximum material composition variation of each quadrant of a material quadtree is smaller than or equal to the preset material composition variation threshold.

The quadtree resolution is used to define the maximum number of levels a quadtree consists of. Normally, the geometric accuracies of a quadtree are determined by the quadtree resolution. If a higher resolution is set, the quadtree can represent the geometries of the original object to a higher level of details.

3.1.3 *Types of Quadrants*

In a quadtree, a quadrant is classified as IN, Partial-IN or OUT. An IN quadrant is completely inside the object while an OUT quadrant is outside. A Partial-IN one is partially inside an object and it is recursively subdivided until a pre-defined quadtree resolution is reached. Similarly, each quadrant in a material quadtree must be classified (equations (2) - (4)). In a material quadtree, four types of quadrant – Void, Homo, Het-In or Het-Bound, are introduced.

1. Void quadrant

In the process of material quadtree generation, some quadrants would be outside the heterogeneous object to be subdivided. These quadrants are considered as containing “void” material and are termed as Void quadrants (Figure 3). These Void quadrants are omitted.

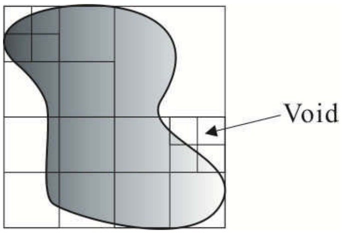


Fig. 3: An example of a Void quadrant.

2. Homo quadrant

Some quadrants are geometrically inside a heterogeneous object. Some of them are classified as Homo quadrant if only a single material is contained inside the quadrants or the maximum material composition variation inside the quadrants is within the preset threshold (Figure 4). The Homo quadrants are not subdivided.

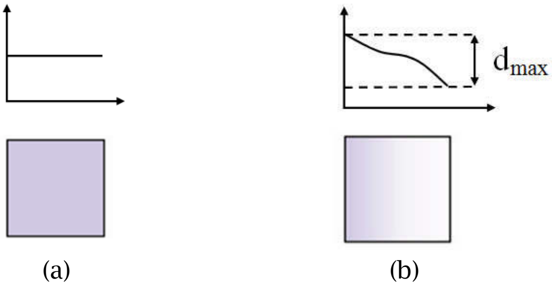


Fig. 4: Examples of Homo quadrants. (a) A homogeneous quadrant. (b) $d_{max} \le \alpha$ in this example quadrant.

3. Het-In quadrant

If the maximum material composition variation inside a quadrant is greater than the preset threshold and the quadrant is inside a heterogeneous object, this quadrant is classified as Het-In (Figure 5).

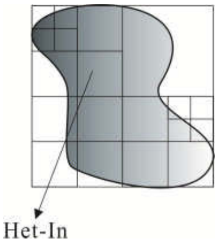


Fig. 5: A Het-In quadrant ($d_{max} > \alpha$).

4. Het-Bound quadrant

If the maximum material composition variation inside a quadrant is greater than the preset threshold and part of the material boundary is inside the quadrant or the quadrant is partially inside the heterogeneous object, the quadrant is classified as a Het-Bound quadrant (Figure 6).

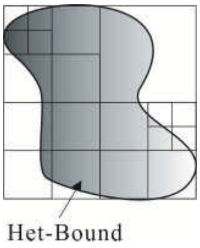


Fig. 6: A Het-Bound quadrant.

For a Het-Bound quadrant, it can be partitioned into a number of cells (called Het-Bound cell) by the part of material boundary inside it (Figure 7). A Het-Bound cell must also be classified and three types of Het-Bound cell are introduced.

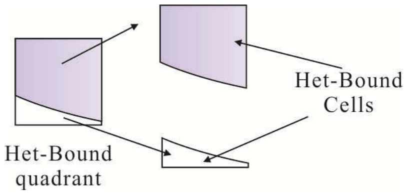


Fig. 7: A Het-Bound quadrant is partitioned into 2 Het-Bound cells.

a. Void-C cell

For this type of cell, “void” material is contained.

b. Homo-C cell

A Het-Bound cell is classified as Homo-C if it is a homogeneous one or the material composition variation inside the cell is within the preset threshold.

c. Het-C cell

Inside this type of cell, the material composition variation is larger than the present threshold.

Based on the definitions of the quadrants and cells of the material quadtree, a material quadtree for a heterogeneous object can be generated.

3.2 Material Quadtree Generation

The material quadtree generation process is similar to a general quadtree generation process, i.e. quadrants are recursively subdivided, except that the criterion for checking whether a quadrant is subdivided is different. To generate a material quadtree, a 2D box enclosing the heterogeneous object is first created. This box is then subdivided into four quadrants and the quadrants are classified accordingly. Proper operations (preserve, omit or recursively subdivide a quadrant) are then performed based on the following criteria:

1. If a quadrant is classified as Void, it is omitted in the material quadtree.
2. If a quadrant is a Homo one, there is no need to further subdivide it and it is preserved in the material quadtree.
3. A Het-In quadrant is further subdivided and the newly formed quadrants will be classified as either Homo or Het-In. The subdivision process for this Het-In quadrant stops when there are no Het-In quadrants generated from the newly-formed quadrants.

4. For a Het-Bound quadrant, its Het-Bound cells are checked.
   a. If all the Het-Bound cells are classified as Homo-C and Void-C cells and the resolution of this quadrant is larger than or equal to the preset quadtree resolution, this quadrant is not subdivided and is retained in the material quadtree.
   b. If all the Het-Bound cells are classified as Homo-C and Void-C cells but the quadrant's resolution is smaller than the preset quadtree resolution, this quadrant is further subdivided in order to achieve a pre-defined geometric resolution.
   c. If one or more of the cells are classified as Het-C, the quadrant is further subdivided.

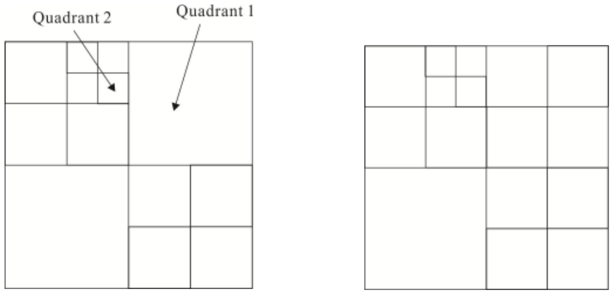
When a Het-Bound quadrant is subdivided, the newly formed quadrants would be classified as Homo, Het-In or Het-Bound. The quadrants are then checked based on the aforementioned criteria and the recursive subdivision process stops when there are no more Het-IN quadrants and Het-Bound quadrants to be subdivided. As the material quadtree will be used for mesh generation in the successive meshing process, Void-C cells of all Het-Bound quadrants are omitted. As a result, the material quadtree of a heterogeneous object is generated and it consists of a number of Homo quadrants and Homo-C cells of Het-Bound quadrants. In Table 1, a comparison between traditional quadtree and the proposed material quadtree is given.

	Material Quadtree	*Quadtree*
Quadrants	Void, Homo, Het-In, Het-Bound	IN, Partial-In, OUT
Cells	Void-C, Homo-C, Het-C	Nil
Quadrants to be subdivided	Het-In and Het-Bound with Het-C cells	Partial-In
Resolution	Quadtree resolution (geometry) Material composition variation threshold (material)	Quadtree resolution (geometry)
Represented by	Homo quadrants and Homo-C cells	IN and Partial-In quadrants

Tab. 1: Comparisons between material quadtree and traditional quadtree.

3.3 Balanced Material Quadtree

Normally, a quadtree is considered as balanced if the sizes of any two neighbor quadrants differ by at most a factor of two. However, the material quadtree formed by the above procedures is normally unbalanced. In the meshing process, a balanced quadtree is preferred. In this case, the material quadtree formed must be balanced. To do this, the quadrants which are unbalanced and the sizes are larger compared with their neighbors are subdivided. This process is repeated until a balanced material quadtree is obtained. In Figure 8, an unbalanced quadtree and its balanced counterpart are shown.



(a) (b)

Fig. 8: An unbalanced quadtree and its balanced counterpart. (a) Quadrant 1 is greater than its neighbor quadrant 2 by four times and this quadtree is an unbalanced one. (b) The quadtree is balanced by subdivided quadrant 1.

3.4 Meshing Process

To generate the mesh of a heterogeneous object, the Homo quadrants and the Homo-C cells of a material quadtree must be triangulated. For a Homo quadrant of a balanced quadtree, there are 6 plausible patterns when considering the number of quadrants next to it and the possible meshes of different patterns are shown in Figure 9.

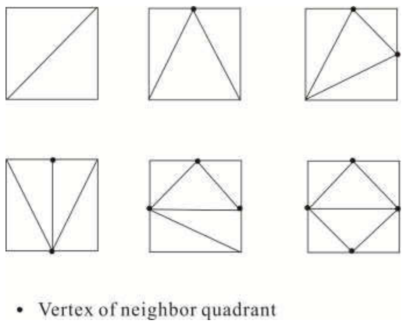


Fig. 9: Six plausible patterns to triangulate a Homo quadrant (the other patterns can be retrieved using the rotational symmetry properties) [6].

The process for triangulating a Homo-C cell is more complicated. In this paper, a Delaunay triangulation is employed. A number of points on the material boundary of the cells are extracted. Normally, more points should be extracted if a higher geometric accuracy is required. These points and the vertices of the Homo-C cell are used as the input of the Delaunay triangulation process and the triangulation of the cell is resulted. As a result, a mesh can be generated from the material quadtree of a heterogeneous object. The quality of some triangular elements may not be acceptable. To improve the quality of the mesh, the mesh should be modified and optimized. In this paper, a mesh vertices relocation process is employed for modifying the mesh. The quality (aspect ratio in this case) of each mesh element is calculated and a vertex is relocated based on the quality of the mesh elements it is on. The process is repeated for all vertices until an acceptable quality is achieved. The details of this process can be found in [6]. Some other mesh modification and optimization methodologies can also be found in [4], [8], [31].

The material composition of each element node can be calculated from the HFT. Together with the mesh, the required materials and geometries of a 2D heterogeneous object can be input to a FEA platform and proper analysis can then be performed.

4 RESULTS

To illustrate the proposed adaptive meshing process, a simple heterogeneous object is used as an example (Figure 10). The number of material the object consists of is assumed to be 3. The material compositions of edge e_4, vertices v_1 and v_2 are (0,1,0), (1,0,0) and (0,0,1) respectively (Equation (1)). Quadratic and exponential distance functions are employed in node H_2 and e_3 respectively. The quadtree resolution is set to 7 and the material composition variation threshold equals to 0.1 in this example. There are 14919 mesh elements in the resultant mesh and the computational time required to generate the mesh is 250 seconds (System information: Windows 7 32-bit system, CPU 2.66GHz, 4GB RAM, CAD platform: UGS NX4.0).

In the first step, a material quadtree is generated (Figure 11). A balanced material quadtree is then created and each quadrant is triangulated. An adaptive mesh for the example 2D heterogeneous object is obtained (Figure 12). To improve the quality of the mesh, the mesh is modified and a modified mesh is shown in Figure 13. In Figure 14, the meshes for different material composition variation thresholds are shown. It can be seen that a finer mesh at the region of greater material composition change is resulted when a smaller threshold is set.

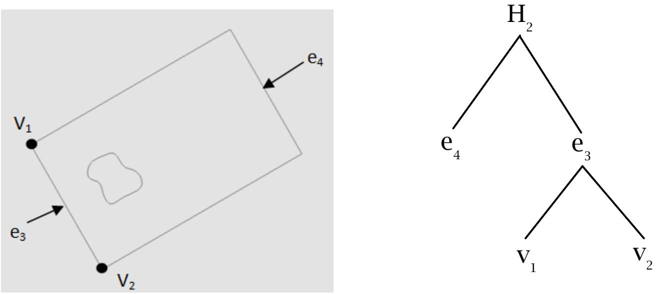
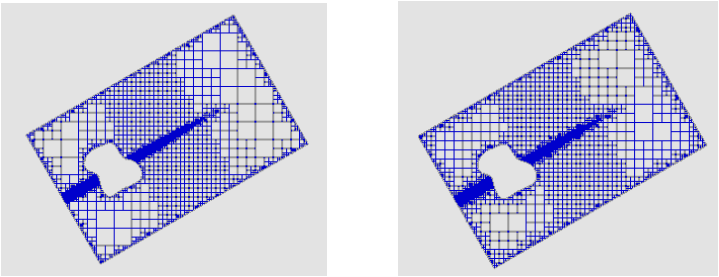


Fig. 10: An example 2D heterogeneous object H_2 and its HFT.



(a) (b)

Fig. 11: (a) Material quadtree of H_2. (b) The balanced counterpart.

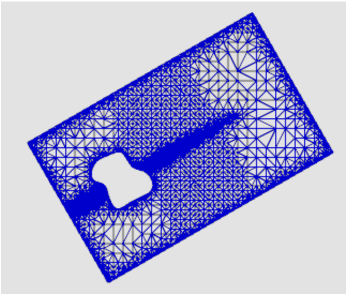


Fig. 12: Mesh of H_2.

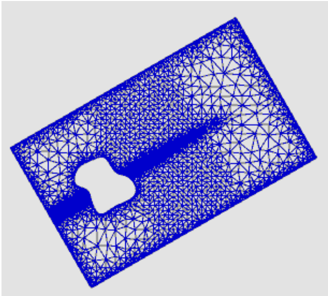
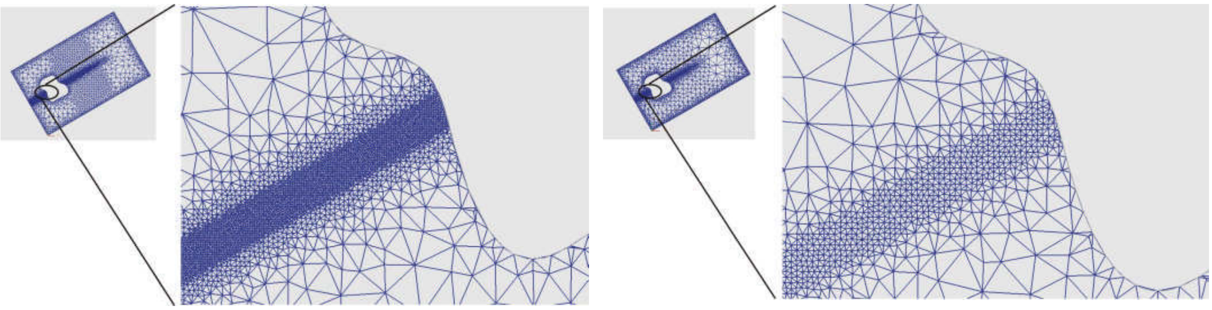


Fig. 13: The mesh after modification



(a) (b)

Fig. 14: (a) Comparison of meshes with different thresholds. (a) Quadtree resolution = 7, material composition variation threshold = 0.1 (b) Quadtree resolution = 7, material composition variation threshold = 0.2

5 DISCUSSION

In this paper, it can be seen that a material quadtree can be used to represent both the geometries and materials of a heterogeneous object to the desired level of details. An adaptive mesh can thus be generated based on this material quadtree. The proposed method is relatively simple to implement. The method can be easily extended to 3D cases, i.e. using octree for generating adaptive mesh of 3D heterogeneous objects.

In the proposed method, there is no need to convert the material representation of a heterogeneous object to another form (e.g. iso-material contours). Only the material compositions of some sampled points inside a quadrant are required to be calculated based on the representation scheme of the object. The maximum "material decomposition difference" of this quadrant is then inferred. For material configurations other than a linear one, this checking process can still be adopted as the material composition of each sampled point can be calculated based on the configuration and so does the maximum "material decomposition difference" of a quadrant. As a result, a heterogeneous object represented by other representation schemes may possibly be adaptively meshed by the proposed method.

Generally, material properties of any points in the object can be inferred from the material compositions of the points. In the material quadtree, the material composition variation inside a quadrant is used as one of the criteria for determining whether a quadrant is subdivided. In the future development of the proposed method, instead of the material composition variation, the material properties variation may be taken into consideration.

In the current mesh modification or optimization process, only geometric compatibility and topological compatibility of the mesh are taken into consideration. In general, when mesh node positions change, the material composition variations inside the corresponding mesh elements may also vary. In some cases, the updated variations of some elements may become larger than the preset threshold value. To account for this problem, the material compatibility may also need to be taken into consideration in the mesh modification or optimization process.

6 CONCLUSION

In this paper, a material quadtree is proposed. Based on this material quadtree, a mesh for a 2D heterogeneous object which is adaptive in both geometries and materials can be generated. As a result, a proper solution for generating a mesh which can guarantee geometric accuracies and characterize the material heterogeneities is provided.

7 ACKNOWLEDGMENTS

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