



## Region Constraint of a Bivariate Rational Interpolating Surface

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### ABSTRACT

This paper focuses on the region constraint of the interpolating surface. The interpolating surface is constructed with a bivariate rational function which is based on the function values and the partial derivatives. If the spline surface protrudes irregularly in a patch, it can be modified by the interpolation parameters. But it is not easy to find the suitable modification parameters. Sometimes the protuberance surface is a big trouble for computer-aided geometric design. To overcome this problem, the sufficient conditions for surface control are derived. The interpolating surface will be above or below a given space plane in the interpolation subinterval by the constraint conditions. Some numerical examples are given to illustrate this method.

**Keywords:** surface constraint, rational interpolation, CAGD.

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### 1 INTRODUCTION

The construction method of curve and surface is a key issue in computer-aided geometric design. There are many ways to tackle this problem, for example, the polynomial spline triangular spline,  $\beta$ -spline, Box spline, vertex spline and others [1-8]. These methods are effective and applied widely in shape design of industrial products. On the other hand, one of the disadvantages of the spline method is its uniqueness; it is not possible to do local modifications while leaving invariant interpolating data. In recent years, the univariate rational spline interpolation with parameters has been constructed. Generally speaking, most of the polynomial spline methods are the interpolating methods[9-15]. Those kinds of interpolation spline not only have simple mathematical representation, but they can be used for the modification of local curves by selecting suitable parameters under the condition that the interpolating data are not changed. In this case, the uniqueness of the interpolating curves for the given interpolating data becomes the uniqueness of the interpolating curves for the given interpolating data and the parameters.

Motivated by the univariate rational spline interpolation, the bivariate rational interpolation spline with parameters, based on the function values and the partial derivative values, has been constructed [16]. The point control, shape control and convexity control of the interpolating surface interval have

been studied[17-19]. In fact, many shape design of industrial products need region constraint. If the surface protrudes irregularly in a patch, It is not easy to fined the suitable parameter to modify. Now we change our eye's sight to another way. Under which conditions will there be no protuberance in the surface? Based on this, the paper deals with the region constraint method of the interpolating surface.

The paper is arranged as follows. In Section 2, the rational interpolation based on the function values and partial derivatives will be restated briefly. Section 3 deals with the base of the interpolation function and a Lemma is derived. The region constraint method is derived in Section 4. Section 5 is a numerical example to given, and it shows the surface in a region must be above or below a given space plane with the interpolating data satisfied the sufficient conditions.

## 2 INTERPOLATION

Let  $\Omega: [a, b; c, d]$  be the plane region, and  $\{(x_i, y_j, f_{i,j}, \frac{\partial f_{i,j}}{\partial x}, \frac{\partial f_{i,j}}{\partial y}), i=1, 2, \dots, n; j=1, 2, \dots, m\}$  be a given set of data points, where  $a = x_1 < x_2 < \dots < x_n = b$  and  $c = y_1 < y_2 < \dots < y_m = d$  are the knot spacings,  $f_{i,j}, \frac{\partial f_{i,j}}{\partial x}, \frac{\partial f_{i,j}}{\partial y}$  represent  $f(x_i, y_j), \frac{\partial f(x, y)}{\partial x}, \frac{\partial f(x, y)}{\partial y}$  at the point  $(x_i, y_j)$  respectively. Let  $h_i = x_{i+1} - x_i, l_j = y_{j+1} - y_j$ , and for any point  $(x, y) \in [x_i, x_{i+1}; y_j, y_{j+1}]$  in the  $xy$  plane, a and let  $\theta = \frac{x-x_i}{h_i}$  and  $\eta = \frac{y-y_j}{l_j}$ . First, for each  $y = y_j, j=1, 2, \dots, m$ , construct the  $x$  direct interpolating curve  $P_{i,j}^*$  in  $[x_i, x_{i+1}]$  [20] this is given by

$$P_{i,j}^*(x) = \frac{P_{i,j}^*(x)}{q_{i,j}^*(x)}, \quad i=1, 2, \dots, n-1, \quad (2.1)$$

Where

$$p_{i,j}^*(x) = (1-\theta)^3 \alpha_{i,j}^* f_{i,j} + \theta(1-\theta)^2 V_{i,j}^* + \theta^2(1-\theta) W_{i,j}^* + \theta^3 \beta_{i,j}^* f_{i+1,j},$$

$$q_{i,j}^*(x) = (1-\theta) \alpha_{i,j}^* + \theta \beta_{i,j}^*,$$

and

$$V_{i,j}^* = (2\alpha_{i,j}^* + \beta_{i,j}^*) f_{i,j} + h_i \alpha_{i,j}^* \frac{\partial f_{i,j}}{\partial x},$$

$$W_{i,j}^* = (\alpha_{i,j}^* + 2\beta_{i,j}^*) f_{i+1,j} - h_i \beta_{i,j}^* \frac{\partial f_{i+1,j}}{\partial x},$$

with  $\alpha_{i,j}^* > 0, \beta_{i,j}^* > 0$ . This interpolation is called the rational cubic interpolation based on function and derivatives which satisfies

$$P_{i,j}^*(x_i) = f_{i,j}, P_{i,j}^*(x_{i+1}) = f_{i+1,j}, P_{i,j}^*(x_i) = \frac{\partial f_{i,j}}{\partial x}, P_{i,j}^*(x_{i+1}) = \frac{\partial f_{i+1,j}}{\partial x}.$$

Obviously, the interpolation function  $P_{i,j}^*$  on  $[x_i, x_{i+1}]$  is unique for the given data  $\{x_r, f_{r,j}, \frac{\partial f_{r,j}}{\partial x}, r=i, i+1\}$  and positive parameters  $\alpha_{i,j}^*, \beta_{i,j}^*$ .

Using the  $x$  direction interpolation function,  $P_{i,j}^*(x), i=1, 2, \dots, n-1; j=1, 2, \dots, m$  defines the bivariate rational interpolation function in  $[x_1, x_n; y_1, y_m]$ . For each pair  $(i, j), i=1, 2, \dots, n-1$  and  $j=1, 2, \dots, m-1$ , let  $\alpha_{i,j} > 0, \beta_{i,j} > 0$ , define the bivariate interpolating function  $P_{i,j}(x, y)$  on  $[x_i, x_{i+1}; y_j, y_{j+1}]$  as follows:

$$P_{i,j}(x,y) = \frac{p_{i,j}(x,y)}{q_{i,j}(x,y)}, \quad i=1,2,\dots,n-1; \quad j=1,2,\dots,m-1, \quad (2.2)$$

where

$$\begin{aligned} p_{i,j}(x,y) &= (1-\eta)^3 \alpha_{i,j} P_{i,j}^*(x) + \eta(1-\eta)^2 V_{i,j} + \eta^2(1-\eta) W_{i,j} + \eta^3 \beta_{i,j} P_{i,j+1}^*(x), \\ q_{i,j}(x,y) &= (1-\eta) \alpha_{i,j} + \theta \beta_{i,j}, \end{aligned}$$

and

$$\begin{aligned} V_{i,j} &= (2\alpha_{i,j} + \beta_{i,j}) P_{i,j}^*(x) + l_j \alpha_{i,j} f_{i,j}^*(x, y_j), \\ W_{i,j} &= (\alpha_{i,j} + 2\beta_{i,j}) P_{i,j+1}^*(x) - l_j \beta_{i,j} f_{i,j+1}^*(x, y_{j+1}), \end{aligned}$$

with

$$f_{r,s}^*(x, y_s) = (1-\theta) \frac{\partial f_{i,s}}{\partial y} + \theta \frac{\partial f_{i+1,s}}{\partial y}, \quad \theta \in [0,1], \quad s = j, j+1. \quad (2.3)$$

It is obvious that  $f_{r,s}^*(x_r, y_s) = \frac{\partial f_{i,s}}{\partial y}$ ,  $r = i, i+1$ ,  $s = j, j+1$ .

The term  $P_{i,j}(x,y)$  is called the bivariate rational interpolation based on function values and partial derivative values which satisfies

$$P_{i,j}(x_r, y_s) = f(x_r, y_s), \quad \frac{\partial P_{i,j}(x_r, y_s)}{\partial x} = \frac{\partial f_{r,s}}{\partial x}, \quad \frac{\partial P_{i,j}(x_r, y_s)}{\partial y} = \frac{\partial f_{r,s}}{\partial y} \quad r = i, i+1, \quad s = j, j+1.$$

It is easy to understand that this form of the interpolating function  $P_{i,j}(x,y)$  on  $[x_i, x_{i+1}; y_j, y_{j+1}]$  is unique for the given data  $(x_r, y_s, f_{r,s}, \frac{\partial f_{r,s}}{\partial x}, \frac{\partial f_{r,s}}{\partial y}, r = i, i+1, s = j, j+1)$  and parameters  $\alpha_{i,j}^*, \beta_{i,j}^*, \alpha_{i,j+1}^*, \beta_{i,j+1}^*$  and  $\alpha_{i,j}, \beta_{i,j}$ .

### 3 THE BASES OF THE INTERPOLATION

From Eqs. (2.1)-(2.3), the interpolating function  $P_{i,j}(x,y)$  defined by(2) can be written as follows:

$$P_{i,j}(x,y) = \sum_{r=i}^{i+1} \sum_{s=j}^{j+1} [a_{r,s}(\theta, \eta) f_{r,s} + b_{r,s}(\theta, \eta) h_i \frac{f_{r,s}}{\partial x} + c_{r,s}(\theta, \eta) l_j \frac{f_{r,s}}{\partial y}], \quad (3.1)$$

where

$$\begin{aligned} a_{i,j}(\theta, \eta) &= \frac{(1-\theta)^2((1+\theta)\alpha_{i,j}^* + \theta\beta_{i,j}^*)}{((1-\theta)\alpha_{i,j}^* + \theta\beta_{i,j}^*)} \bullet \frac{(1-\eta)^2((1+\eta)\alpha_{i,j} + \theta\beta_{i,j})}{((1-\eta)\alpha_{i,j} + \eta\beta_{i,j})} \\ a_{i+1,j}(\theta, \eta) &= \frac{\theta^2((1-\theta)\alpha_{i,j}^* + (2-\theta)\beta_{i,j}^*)}{((1-\theta)\alpha_{i,j}^* + \theta\beta_{i,j}^*)} \bullet \frac{(1-\eta)^2((1+\eta)\alpha_{i,j} + \theta\beta_{i,j})}{((1-\eta)\alpha_{i,j} + \eta\beta_{i,j})} \\ a_{i,j+1}(\theta, \eta) &= \frac{(1-\theta)^2((1+\theta)\alpha_{i,j+1}^* + \theta\beta_{i,j+1}^*)}{((1-\theta)\alpha_{i,j+1}^* + \theta\beta_{i,j+1}^*)} \bullet \frac{\eta^2((1-\eta)\alpha_{i,j} + (2-\theta)\beta_{i,j})}{((1-\eta)\alpha_{i,j} + \eta\beta_{i,j})} \\ a_{i+1,j+1}(\theta, \eta) &= \frac{\theta^2((1-\theta)\alpha_{i,j+1}^* + (2-\theta)\beta_{i,j+1}^*)}{((1-\theta)\alpha_{i,j+1}^* + \theta\beta_{i,j+1}^*)} \bullet \frac{\eta^2((1-\eta)\alpha_{i,j} + (2-\theta)\beta_{i,j})}{((1-\eta)\alpha_{i,j} + \eta\beta_{i,j})} \\ b_{i,j}(\theta, \eta) &= \frac{\theta(1-\theta)^2 \alpha_{i,j}^*}{((1-\theta)\alpha_{i,j}^* + \theta\beta_{i,j}^*)} \bullet \frac{(1-\eta)^2((1+\eta)\alpha_{i,j} + \eta\beta_{i,j})}{((1-\eta)\alpha_{i,j} + \eta\beta_{i,j})} \end{aligned}$$

$$\begin{aligned}
b_{i+1,j}(\theta,\eta) &= \frac{-\theta^2(1-\theta)\beta_{i,j}^*}{((1-\theta)\alpha_{i,j}^* + \theta\beta_{i,j}^*)} \bullet \frac{(1-\eta)^2((1+\eta)\alpha_{i,j} + \eta)\beta_{i,j}}{(1-\eta)\alpha_{i,j} + \eta\beta_{i,j}} \\
b_{i,j+1}(\theta,\eta) &= \frac{\theta(1-\theta)^2\alpha_{i,j+1}^*}{((1-\theta)\alpha_{i,j}^* + \theta\beta_{i,j}^*)} \bullet \frac{\eta^2((1-\eta)\alpha_{i,j} + (2-\eta)\beta_{i,j})}{((1-\eta)\alpha_{i,j} + \eta\beta_{i,j})} \\
b_{i+1,j+1}(\theta,\eta) &= \frac{-\theta^2(1-\theta)\beta_{i,j+1}^*}{((1-\theta)\alpha_{i,j}^* + \theta\beta_{i,j}^*)} \bullet \frac{\eta^2((1-\eta)\alpha_{i,j} + (2-\eta)\beta_{i,j})}{((1-\eta)\alpha_{i,j} + \eta\beta_{i,j})} \\
c_{i,j}(\theta,\eta) &= (1-\theta)\alpha_{i,j} \bullet \frac{\eta(1-\eta)^2}{((1-\eta)\alpha_{i,j} + \eta\beta_{i,j})} \\
c_{i+1,j}(\theta,\eta) &= \theta\alpha_{i,j} \bullet \frac{\eta(1-\eta)^2}{((1-\eta)\alpha_{i,j} + \eta\beta_{i,j})} \\
c_{i,j+1}(\theta,\eta) &= -(1-\theta)\beta_{i,j} \bullet \frac{\eta^2(1-\eta)}{((1-\eta)\alpha_{i,j} + \eta\beta_{i,j})} \\
c_{i+1,j+1}(\theta,\eta) &= -\theta\beta_{i,j} \bullet \frac{\eta^2(1-\eta)}{((1-\eta)\alpha_{i,j} + \eta\beta_{i,j})}
\end{aligned}$$

**Lemma** If  $P_{i,j}(x,y)$  is the interpolation defined by (2.2), no matter what positive number the parameters  $\alpha_{i,j}, \beta_{i,j}, \alpha_{i,j}^*, \beta_{i,j}^*$  and  $\alpha_{i+1,j}^*, \beta_{i+1,j}^*$  take, the bivariate spline bases satisfy

$$\begin{aligned}
a_{i,j}(\theta,\eta) + a_{i+1,j}(\theta,\eta) + a_{i,j+1}(\theta,\eta) + a_{i+1,j+1}(\theta,\eta) &= 1, \\
b_{i,j}(\theta,\eta) - b_{i+1,j}(\theta,\eta) + b_{i,j+1}(\theta,\eta) - b_{i+1,j+1}(\theta,\eta) &= \theta(1-\theta), \\
c_{i,j}(\theta,\eta) + c_{i+1,j}(\theta,\eta) - c_{i,j+1}(\theta,\eta) - c_{i+1,j+1}(\theta,\eta) &= \eta(1-\eta).
\end{aligned}$$

#### 4 GLOBAL CONSTRAINT

For the interpolation that defined by (2.2), When the interpolating data satisfy some conditions, the surface in a subregion must be above or below a given space plane, the conditions are described by the following Theorem 1 and Theorem 2.

**Theorem1** For the given interpolation data  $x_r, y_s, f_{r,s}, \frac{\partial f_{r,s}}{\partial x}, \frac{\partial f_{r,s}}{\partial y}, r = i, i+1, s = j, j+1$ ,  $P_{i,j}(x,y)$  is the interpolation function in  $[x_i, x_{i+1}; y_j, y_{j+1}]$  defined by (2.2),  $M$  is a positive constant, if

$$f_{r,s} < M, \quad \left| h_r \frac{\partial f_{r,s}}{\partial x} \right| < M - f_{r,s}, \quad \left| l_s \frac{\partial f_{r,s}}{\partial y} \right| < M - f_{r,s},$$

and

$$\max_{r,s} (M - f_{r,s}) < 2 \min_{r,s} (M - f_{r,s}) \quad r = i, i+1; s = j, j+1,$$

then  $P_{i,j}(x,y) < M$  for all  $(x,y) \in [x_i, x_{i+1}; y_j, y_{j+1}]$ .

**Proof** From Lemma 1

$$a_{i,j}(\theta,\eta) + a_{i+1,j}(\theta,\eta) + a_{i,j+1}(\theta,\eta) + a_{i+1,j+1}(\theta,\eta) = 1,$$

then

$$P_{i,j}(x,y) < M$$

equals to

$$\sum_{r=i}^{i+1} \sum_{s=j}^{j+1} [a_{r,s}(\theta, \eta)(f_{r,s} - M) + b_{r,s}(\theta, \eta)h_r \frac{f_{r,s}}{\partial x} + c_{r,s}(\theta, \eta)l_j \frac{f_{r,s}}{\partial y}] < 0. \tag{4.1}$$

Thus, if the following inequality (4.2) holds, then (4.1) holds.

$$\begin{aligned} & a_{i,j}(\theta, \eta)(f_{i,j} - M) + a_{i+1,j}(\theta, \eta)(f_{i+1,j} - M) + a_{i,j+1}(\theta, \eta)(f_{i,j+1} - M) + a_{i+1,j+1}(\theta, \eta)(f_{i+1,j+1} - M) \\ & + b_{i,j}(\theta, \eta)(M - f_{i,j}) - b_{i+1,j}(\theta, \eta)(M - f_{i+1,j}) + b_{i,j+1}(\theta, \eta)(M - f_{i,j+1}) + b_{i+1,j+1}(\theta, \eta)(M - f_{i+1,j+1}) \\ & + c_{i,j}(\theta, \eta)(M - f_{i,j}) + c_{i+1,j}(\theta, \eta)(M - f_{i+1,j}) - c_{i,j+1}(\theta, \eta)(M - f_{i,j+1}) - c_{i+1,j+1}(\theta, \eta)(M - f_{i+1,j+1}) < 0 \end{aligned} \tag{4.2}$$

Since

$$a_{r,s}(\theta, \eta) > 0, \quad r = i, i + 1; s = j, j + 1,$$

$$b_{i,j}(\theta, \eta) > 0, b_{i+1,j}(\theta, \eta) < 0, b_{i,j+1}(\theta, \eta) > 0, b_{i+1,j+1}(\theta, \eta) < 0,$$

$$c_{i,j}(\theta, \eta) > 0, c_{i+1,j}(\theta, \eta) > 0, c_{i,j+1}(\theta, \eta) < 0, c_{i+1,j+1}(\theta, \eta) < 0,$$

the following inequality (4.3) holds, then (4.2) holds

$$\begin{aligned} & -(a_{i,j}(\theta, \eta) + a_{i+1,j}(\theta, \eta) + a_{i,j+1}(\theta, \eta) + a_{i+1,j+1}(\theta, \eta)) \min_{r,s} (M - f_{r,s}) \\ & + (b_{i,j}(\theta, \eta) - b_{i+1,j}(\theta, \eta) + b_{i,j+1}(\theta, \eta) + b_{i+1,j+1}(\theta, \eta)) \max_{r,s} (M - f_{r,s}) \\ & + (c_{i,j}(\theta, \eta) + c_{i+1,j}(\theta, \eta) - c_{i,j+1}(\theta, \eta) - c_{i+1,j+1}(\theta, \eta)) \max_{r,s} (M - f_{r,s}) < 0. \end{aligned} \tag{4.3}$$

From Lemmal

$$b_{i,j}(\theta, \eta) - b_{i+1,j}(\theta, \eta) + b_{i,j+1}(\theta, \eta) - b_{i+1,j+1}(\theta, \eta) = \theta(1 - \theta),$$

$$c_{i,j}(\theta, \eta) + c_{i+1,j}(\theta, \eta) - c_{i,j+1}(\theta, \eta) - c_{i+1,j+1}(\theta, \eta) = \eta(1 - \eta),$$

the inequality (4.3) equals to following inequality

$$-\min_{r,s} (M - f_{r,s}) + \theta(1 - \theta) \max_{r,s} (M - f_{r,s}) + \eta(1 - \eta) \max_{r,s} (M - f_{r,s}) < 0 \tag{4.4}$$

From  $\theta \in (0,1)$  and  $\eta \in (0,1)$ , thus

$$\max_{r,s} (\theta(1 - \theta) + \eta(1 - \eta)) = \frac{1}{2}.$$

Thus if following inequality holds, the inequality (4.4) holds

$$-\min_{r,s} (M - f_{r,s}) + \frac{1}{2} \max_{r,s} (M - f_{r,s}) < 0. \tag{4.5}$$

The proof completes.

Similarly, the sufficient condition for the interpolating surface to below a space plane is described by Theorem 2.

**Theorem 2** For the given interpolation data  $x_r, y_s, f_{r,s}, \frac{\partial f_{r,s}}{\partial x}, \frac{\partial f_{r,s}}{\partial y}, r = i, i + 1, s = j, j + 1, P_{i,j}(x, y)$  is the interpolation function in  $[x_i, x_{i+1}; y_j, y_{j+1}]$  defined by (2.2),  $N$  is a positive constant, if

$$f_{r,s} > N, \quad \left| h_r \frac{\partial f_{r,s}}{\partial x} \right| < f_{r,s} - N, \quad \left| l_s \frac{\partial f_{r,s}}{\partial y} \right| < f_{r,s} - N,$$

and

$$\max_{r,s} (f_{r,s} - N) < 2 \min_{r,s} (f_{r,s} - N) \quad r = i, i + 1; s = j, j + 1,$$

then  $P_{i,j}(x, y) > N$  for all  $(x, y) \in [x_i, x_{i+1}; y_j, y_{j+1}]$ .

**Theorem 3** For the given interpolation data  $x_r, y_s, f_{r,s}, \frac{\partial f_{r,s}}{\partial x}, \frac{\partial f_{r,s}}{\partial y}, r = i, i+1, s = j, j+1$ ,  $P_{i,j}(x, y)$  is the interpolation function in  $[x_i, x_{i+1}; y_j, y_{j+1}]$  defined by (2.2),  $M, N$  is a positive constant, if

$$N < f_{r,s} < M, \quad f_{r,s} - N < \max_{r,s} \left| h_r \frac{\partial f_{r,s}}{\partial x} \right|, \left| l_s \frac{\partial f_{r,s}}{\partial y} \right| < M - f_{r,s},$$

and  $\max_{r,s} (f_{r,s} - N) < 2 \min_{r,s} (f_{r,s} - N)$  and  $\max_{r,s} (M - f_{r,s}) < 2 \min_{r,s} (M - f_{r,s}) \quad r = i, i+1; s = j, j+1$ ,

then  $N < P_{i,j}(x, y) < M$ . for all  $(x, y) \in [x_i, x_{i+1}; y_j, y_{j+1}]$ .

**5 NUMERICAL EXAMPLE**

**Example 1.** Assume the function being interpolated,  $f(x, y)$ , is defined on  $[0,1;0,1]$ , and the data are given in the following table 1.

$(x, y)$	(0,0)	(1,0)	(0,1)	(1,1)
$f(x, y)$	4.0	5.5	5.5	5.0
$\frac{\partial f(x, y)}{\partial x}$	2.0	-0.1	0.5	-0.5
$\frac{\partial f(x, y)}{\partial y}$	1.8	0.01	-1.5	-0.01

Tab. 1: The given interpolating data.

For this interpolation, the parameters  $\alpha_{i,j}, \beta_{i,j}, \alpha_{i,j}^*, \beta_{i,j}^*$ , and  $\alpha_{i,j+1}^*, \beta_{i,j+1}^*$  can be any positive real numbers. For instance, let  $\alpha_{i,j} = 2.5, \beta_{i,j} = 4.2, \alpha_{i,j}^* = 0.8, \beta_{i,j}^* = 1.6, \alpha_{i,j+1}^* = 3.8, \beta_{i,j+1}^* = 2.7$  and let  $P(x, y)$  be the interpolating function defined by (2) in  $[0,1;0,1]$  with the data and parameters above. Fig.1 is the graph of the  $P_1(x, y)$ .

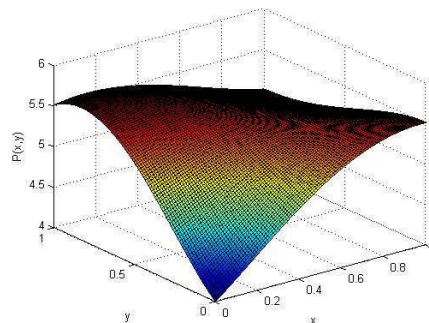


Fig. 1: The graph of interpolating surface  $P_1(x, y)$ .

For the interpolating surface  $P_1(x, y)$  can be constrained to be “down” or “up” by the selected plane in the subinterval. From theorem1 and theorem2, the constraint constant  $M = 7.2$  and  $N = 2.4$  can be selected respectively. Fig. 2 and Fig. 3 are the graphs of the region constraint in  $[0,1;0,1]$ . Without loss of generality, the parameters can be selected by others positive real numbers. For instance, let  $\alpha_{i,j} = 0.2, \beta_{i,j} = 3.2, \alpha_{i,j}^* = 4.8, \beta_{i,j}^* = 3.6$ , and  $\alpha_{i,j+1}^* = 1.5, \beta_{i,j+1}^* = 2.4$ . Fig.4 is the graph of the surface constraint by  $M = 7.2$  and  $N = 2.4$ .

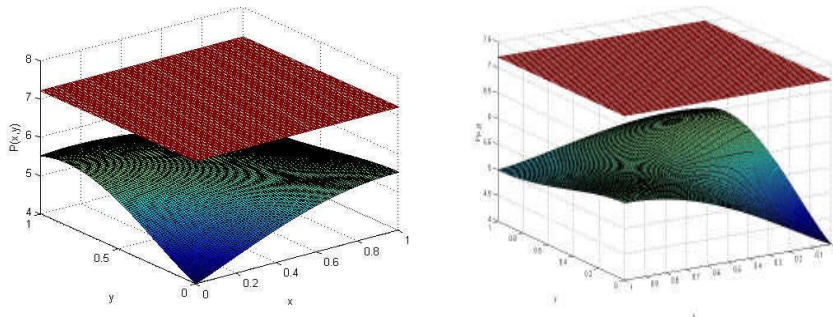


Fig. 2: The graph of the surface constraint with  $M = 7.2$ . (a) standard view (b) different view.

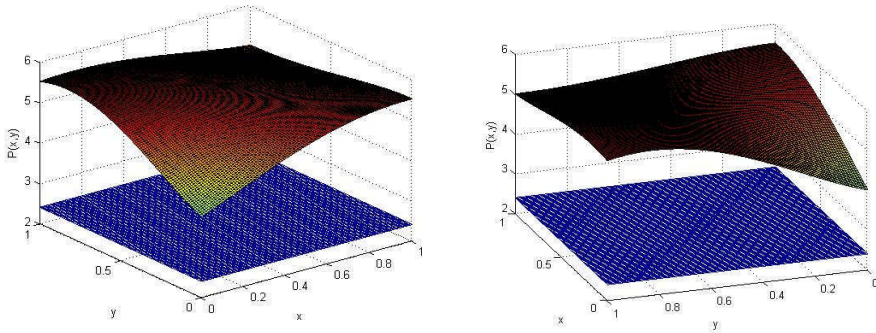


Fig. 3: The graph of the surface constraint with  $N = 2.4$ . (a) standard view (b) different view.

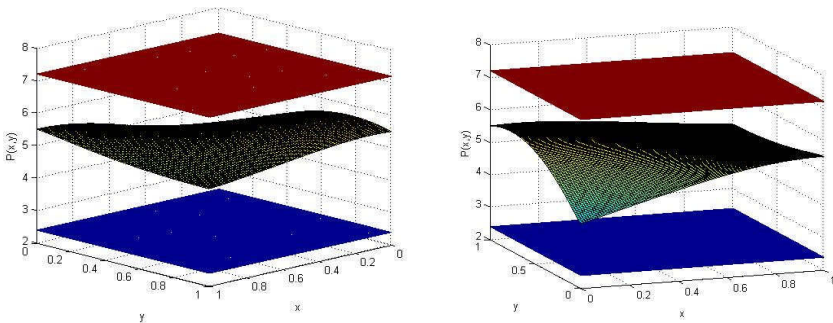


Fig. 4: The graph constraint with different parameters. (a) standard view (b) different view.

**Example 2.** As in Example 1, also let  $[0,1;0,1]$  be the plane region and let the interpolation data be given in Table 2.

$(x, y)$	(0,0)	(1,0)	(0,1)	(1,1)
$f(x, y)$	2.8	3.0	3.2	2.5
$\frac{\partial f(x, y)}{\partial x}$	-0.5	0.3	-0.8	-0.2
$\frac{\partial f(x, y)}{\partial y}$	-0.4	-0.6	1.0	-0.2

Tab. 2: The given interpolating data.

From theorem1 and Theorem 2, the constraint constant  $M = 6.0$  and  $N = 1.3$  can be selected respectively. For instance, let  $\alpha_{i,j} = 3.5, \beta_{i,j} = 2.0, \alpha_{i,j}^* = 1.8, \beta_{i,j}^* = 0.6$ , and  $\alpha_{i,j+1}^* = 1.5, \beta_{i,j+1}^* = 0.5$ . Fig. 5 is the graph of the  $P_2(x, y)$ . If the different parameters are selected  $\alpha_{i,j} = 0.2, \beta_{i,j} = 1.5, \alpha_{i,j}^* = 0.8, \beta_{i,j}^* = 3.2$ , and  $\alpha_{i,j+1}^* = 0.5, \beta_{i,j+1}^* = 3.0$ , Fig. 6 is the graph of the  $P_3(x, y)$ . Fig. 7 is the graph of the surface constraint by  $M = 7.2$  and  $N = 2.4$ . Fig. 8 is the graph of the surface constraint with different view. From the figures, the surface constraint keeps invariant with different interpolation data and different parameters.

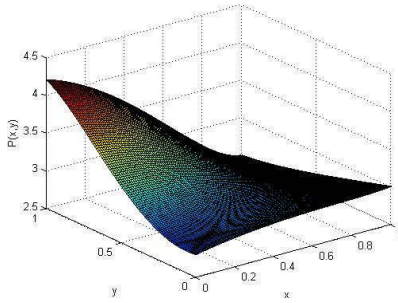


Fig. 5: The graph of surface  $P_2(x, y)$ .

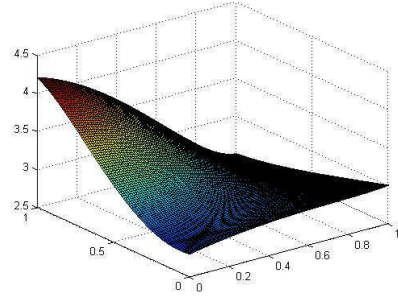


Fig. 6: The graph of surface  $P_3(x, y)$ .

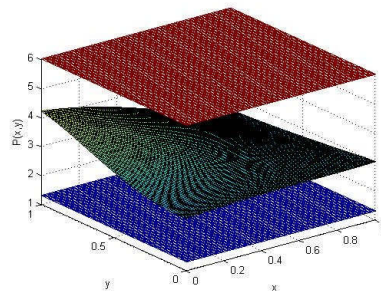
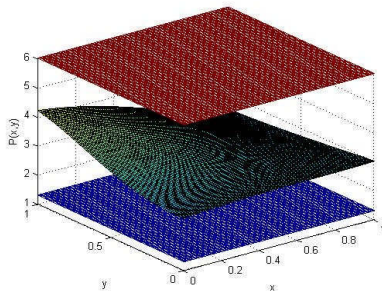


Fig. 7: The graph of the surface constraint with  $M = 6.0, N = 1.3$ . (a) surface  $P_2(x, y)$  (b) surface  $P_3(x, y)$ .

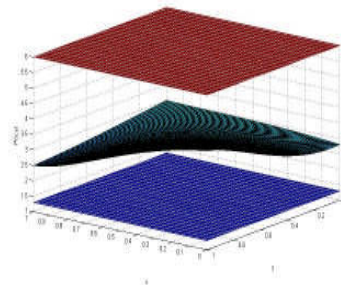
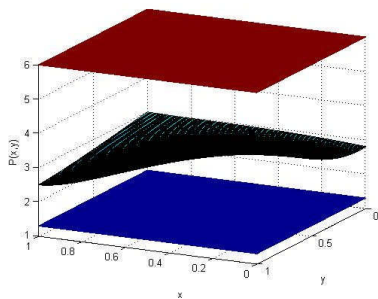


Fig. 8: The graph surface constraint with different view. (a) surface  $P_2(x, y)$  (b) surface  $P_3(x, y)$ .

## 6 CONCLUSIONS

The work in this paper is a contribution towards the region constraint of the bivariate interpolating surface. The sufficient conditions are derived. When the interpolation data are satisfied the constraint conditions, the surface can be controlled to be “down” or “up” with a given plane. The plane is selected also through the constraint condition. The interpolating surface can be constrained in the



subinterval. Under the conditions there will be no protuberance in the surface. There are still several questions that remain unclear. Because of the complexity of the interpolation surface with six parameters, it is very difficult to select the optimal constraint plane to control the interpolating surface. However, because the conditions are inequalities, the relative optimal constraint plane can be derived automatically just by a computer program. For all that, the conditions presented in this paper can provide a way to solve the surface protrudes problem. The numerical experiments show the effectiveness in surface design.

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