Conjugate Curve Design with Spline Contact Path

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ABSTRACT

The paper represents the first effort that extends the free-form geometric modeling technique to conjugation design between surfaces and tooth profile synthesis in gearing. Traditionally, conjugate profiles in gearing are limited to specific curves or surfaces such as involutes that are known satisfying the conjugation condition. The difficulty of a conjugation design lies on the coupling of two conjugate surfaces. This paper introduces spline conjugation and thus offers unlimited freedom and flexibility to the design of conjugate tooth profiles that allows one to pursue profile optimization for the best performance. A general model is introduced to describe the relationship between conjugate curves via contact path and sufficient conditions are obtained to justify non-undercutting. The use of splines in contact path offers the versatility of free-form surface modeling to conjugation design while maintains the traditional involutes as a special case. The paper also demonstrates how control points of B-splines can be used to simplify the design validation process. In gearing, the proposed method may effectively liberates tooth profiles from the traditional conjugate curves and open up conjugation design to pursue optimal conjugation, which is especially important for increasing power density in large gears.

Keywords: conjugation, B-splines, gear, CAD.
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1 INTRODUCTION

Conjugation is a key consideration in contact mechanics for mechanical devices such as gears and pumps. These mechanisms involve a pair of contact surfaces attached to two coupled rigid bodies, which move against each other along contact traces (Fig. 1). Conjugation design mainly refers to the design of a pair of contact surfaces. A good conjugation may lead to smooth motion, high load endurance, high efficiency, and low transmission errors.

In gearing, the conjugate surfaces are the tooth profiles. The planar conjugation can be modeled by curves in the rotation planes. Currently these conjugate curves are selected or modeled by a small pool of traditional curves and their combinations [6],[9],[14],[14]. Among them, involute profiles are the most popular and the industry standard [2],[5]. However, involute is chosen for manufacturability and exchangeability rather than performance—contact stress, bending stress, transmission error, efficiency, etc. When high performance is in demand, curves must be modified. Many researches are focused on creating new conjugate geometries. Splines could be a powerful tool for conjugation design. But it is very difficult to design free-form spline curves for conjugation because the two coupled conjugate curves must satisfy the conjugate condition without any cusp (or undercutting in gearing). This paper originally presents a systematic study for conjugation design with splines. The idea of a single free-form curve design extends to a pair of coupling curves design.

Splines are widely used in computer aided design (CAD) and computer aided manufacturing (CAM). Bezier, B-spline and NURBS are the most popular and standard in industry [11]. These splines have been applied in mechanisms including centrodes of non-circular gears and pumps as well as cam motion programs [7],[10],[12]. The design flexibility and freedoms can be enhanced dramatically with splines. Theoretically, splines can be used to interpolate and approximate any geometric shapes. Conjugation design is a kind of coupled surfaces design. The issue with conjugation design is how to generate two surfaces with desirable contact properties. If a proper model can be established for conjugation design, splines may offer the freedom and flexibility in conjugation design for performance. Because of the importance of gears in machinery, the application of this paper is focused on gear conjugation design. The results can be applied to any planar conjugation. Generally, conjugation and non-undercutting are the two conditions of a conjugation leading to a smooth transmission.

For two rigid bodies moving relative to each other, a curve on a rigid body corresponds to a conjugate curve on the other body. With this concept, one may select a curve and find the matching curve. However, this is difficult in conjugation design because of the coupling between curves. In a typical situation, a transmission occurs between two moving bodies with a reference frame. The contact point of the conjugate curve traces a contact path on the frame and the contact path reflects the properties of the contact. Each given contact path corresponds to a pair of conjugate curves. In other words, one may design or search for the optimal spline-based contact path and then identify the corresponding conjugate curves.

The paper will offer a brief review on the basic concepts of centrodes and conjugation in gearing and presents the conjugation and undercutting equations for conjugation design via contact path. Conditions are obtained for general parametric contact path. Splines are used for gear conjugation design. Examples showing the use of B-splines in contact path are presented.

2 CONTACT CURVES, CUTTER PROFILE, AND CONTACT

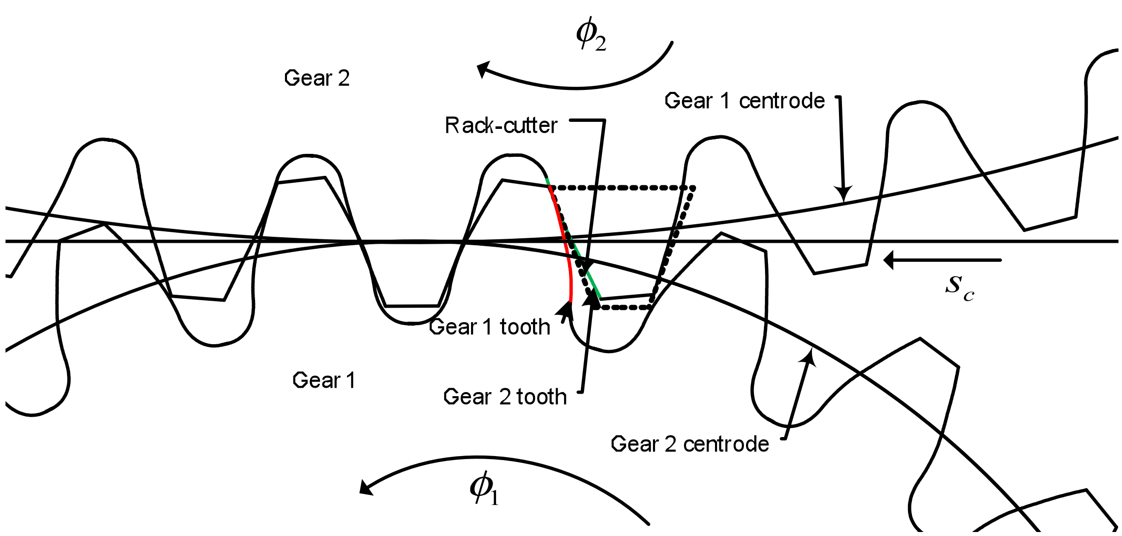


Fig. 1: Gearing contact: Gears 1 and 2 rotate in opposite directions and the rack-cutter translates to the left.

For any two rigid bodies moving relative to each other, there exists an instant center at any instant. On both bodies, the points at the instant center have zero relative velocity. The locus of the instant center on each body is called centrodes. The relative motion between the two bodies is equivalent to the pure rolling between their centrodes. At any instant, the centrodes share a common tangent at the instant center. In Fig. 1, if gear 1 is fixed, as gear 2 rotates around gear 1, the centrode on gear 2 rolls along the centrode on gear 1 without slipping.

Transmission by rolling contact is made possible by friction and its application is limited to low speed and torque. For high speed or high torque transmission, the transmission may be accomplished by the contact between conjugate profiles, such as the tooth profiles in Fig. 1. The conjugate profiles roll and slide at the contact points. For two rigid bodies moving relative to each other, there are infinite number of conjugate profiles, which are capable of transmitting the same motion. The centrodes are a special conjugate pair with rolling contact. The paper will offer the theoretical foundation for finding a suitable pair of conjugate profiles for a satisfactory performance.

As mentioned before, for a relative motion between two bodies, one curve may be assumed and its conjugate curve on the other body can then be meshed. However, a general relative motion usually makes constraints nonlinear and the treatment difficult. Furthermore, splines are not accurate for standard involute tooth geometry representation [1]. A different curve must be found for a general representation which can include at least the most popular gearing geometry—involute profiles.

On the fixed frame, the trace of contact points between a pair of conjugate curves is the contact path. Each pair of conjugate curves corresponds to a contact path. For involute contact curves, the contact path is a straight line segment [8]. Splines are capable to represent straight lines accurately [11]. However, the relationship between the contact path and each of the corresponding conjugate curves is not a simple conjugation.

In planar gearing, a rack cutter may be used to cut the conjugate spur gear tooth profiles. With the rack-cutter taken into consideration, a gearing system involves four rigid bodies: the two rotating bodies with the conjugate profiles, the rack-cutter, and the fixed frame. The rack-cutter is conjugate to each of the gear conjugate profiles and has a straight-line centrode against each gear. The transformation between a cutter and the contact path or fixed frame is a simple translation. Therefore, for a given contact path, the contact gear tooth profiles can be obtained through an intermediate (fictitious) rack-cutter profile and the conjugation between the desired contact profiles can be guaranteed. This would effectively eliminate the difficulty of deriving the constraint conditions for contact path and the conjugate curves can be easily obtained from a given contact path.

3 CONJUGATION AND UNDERCUTTING

To have a smooth transmission through a pair of contact curves, the contact curves must satisfy the conjugation condition and contain no undercutting.

Consider the relative motion between a gear (Fb) and the rack-cutter (Mb) (Fig. 2). The coordinate systems F_f called the fixed system and F_m called the moving system are attached to Fb and Mb respectively. For convenience, Fb is assumed stationary. In the motion of Mb with respect to Fb, let $\vec{c}_f$ in F_f be the fixed centrode and $\vec{c}_m$ in F_m the moving centrode. The motion of Mb with respect to Fb is equivalent to the rolling motion of $\vec{c}_m$ against $\vec{c}_f$. Let I be the instant center, i.e. the contact point of $\vec{c}_m$ and $\vec{c}_f$ at the instant. A coordinate system F_I called instant system (or Ferret frame) is defined such that the origin is at I, the contact point of $\vec{c}_m$ and $\vec{c}_f$, and the x -axis in the tangent direction of $\frac{d\vec{c}_f}{ds_c}$, where s_c is the common rolling arc length of the centrodes. Assume F_m and F_I are coincident initially, i.e. $F_m = F_I$, at $s_c = 0$. The transformation from a point $\begin{pmatrix} x^m \\ y^m \end{pmatrix}$ in F_m to F_I is $\begin{pmatrix} x^I \\ y^I \end{pmatrix} = \begin{pmatrix} x^m \\ y^m \end{pmatrix} - \begin{pmatrix} s_c \\ 0 \end{pmatrix}$.

In the following discussion, let the subscript of a variable refer to the attached body while the superscript the reference system. When the attached system and the referenced system are the same, the superscript is ignored. For example, the position vector of a point I_f^m is referred to a point attached to F_f but referenced in system F_m.

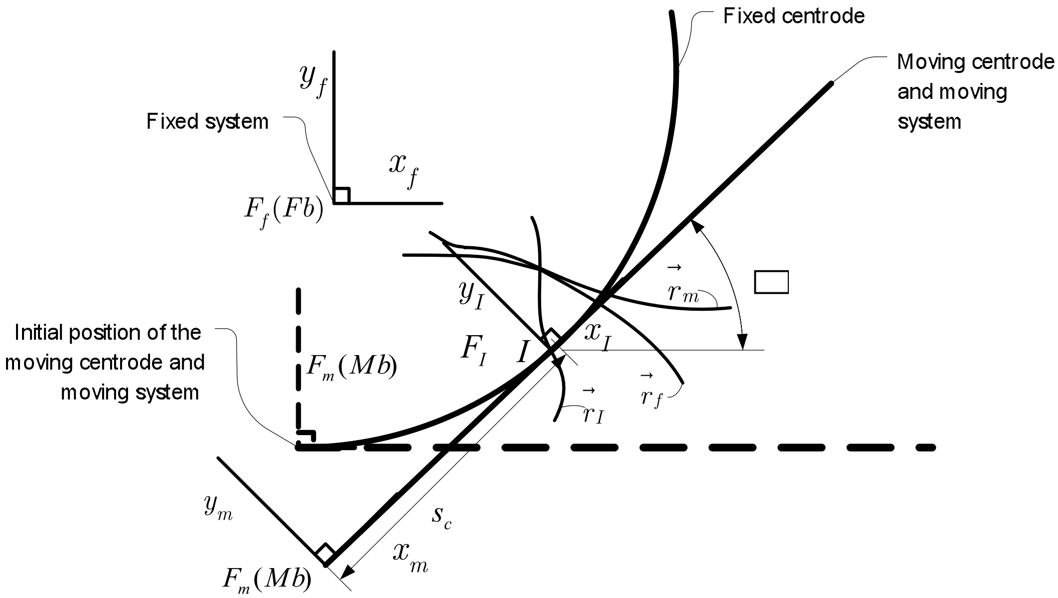


Fig. 2: Fixed system F_f, moving system F_m and instant system (Frenet system) F_I. The straight line is the moving centrode of the rack-cutter, while the long curve is the fixed centrode of a gear. The curves $\vec{r}_f, \vec{r}_I, \vec{r}_m$ are schematically drawn at the instant.

If a curve $\vec{r}_m$ in F_m is conjugate to $\vec{r}_f$ in F_f, then the contact points trace a contact path $\vec{r}_I$ in F_I. There will be rolling and sliding between the contact curves $\vec{r}_m$ and $\vec{r}_f$, and the relationship between them is governed by the conjugate condition, i.e. the conjugated contact curves have a common normal at the contact point [2]. The conjugate condition can be expressed as $\vec{r}_I^{\,l} \bullet d\vec{r}_m^{\,l} = 0$, where the superscript $l = f, m, I$. The expression can be re-written as

$$\begin{pmatrix} x_I^l \\ y_I^l \end{pmatrix} \bullet \begin{pmatrix} dx_m^l \\ dy_m^l \end{pmatrix} = x_I^l dx_m^l + y_I^l dy_m^l = 0. \tag{3.1}$$

This is the equation of conjugation (also called equation of meshing). The relative velocity at a contact point can be expressed by

$$\vec{v}^{\,l} = \vec{v}_f^{\,l} - \vec{v}_m^{\,l} = d\vec{r}_f^{\,l} - d\vec{r}_m^{\,l} = \begin{pmatrix} dx_f^l \\ dy_f^l \end{pmatrix} - \begin{pmatrix} dx_m^l \\ dy_m^l \end{pmatrix} \tag{3.2}$$

where $l = f, m, I$. In F_I , $\vec{v}^{\,l} = \vec{\omega} \times \vec{r}_I = \begin{pmatrix} -y_I \\ x_I \end{pmatrix} d\psi$, where y is the relative angular displacement between Mb and Fb (Fig. 2). Since centrode $\vec{r}_m$ is on the tangent line of $\vec{r}_f$ at any instant, there is a curvature relationship of $\frac{d\psi}{ds_c} = \kappa_c$, where κ_c is the curvature of the fixed centrode $\vec{c}_f$.

A cusp point is a singular point on a curve, where the curvature is infinity. At a cusp, the intended contact between surfaces may begin to occur inside the tooth profile. In other words, undercutting will occur and a portion of tooth surface may be cut off. Therefore, the intended contact becomes impossible. If a cusp occurs on curve $\vec{r}_f$, then with Eqn. (3.2), in system F_I , $dr_f^I = 0$, or - $\vec{v}^{\,I} + d\vec{r}_m^{\,I} = 0$, which yields the equation of cusp (or equation of undercutting),

$$\begin{pmatrix} -y_I \\ x_I \end{pmatrix} d\psi + \begin{pmatrix} dx_m^I \\ dy_m^I \end{pmatrix} = 0 \tag{3.3}$$

To avoid undercutting, a curve must be truncated at a cusp point or connected to another curve. An undercutting-free tooth profile or curve must be free of cusp.

Using the transformation between F_I and F_m , equation of conjugation and equation of cusp can be expressed in F_I. From Eqn. (3.1), the equation of conjugation is

$$x_I ds_c + x_I dx_I + y_I dy_I = 0 \tag{3.4}$$

Equation (3.3) yields

$$-y_I d\psi + dx_I + ds_c = 0 \tag{3.5}$$

$$x_I d\psi + dy_I = 0 \tag{3.6}$$

The common arc length s_c of the centrodes $\vec{c}_m$ and $\vec{c}_f$ can be used as a general motion parameter. The equations above give relationship between $\vec{r}_m$ and the referenced contact path $\vec{r}_I$. On the other hand, $\vec{r}_I$ can be naturally parameterized by the (arc) length parameter s_I according to differential geometry. To use s_c as a parameter for $\vec{r}_I$, there must exist $\frac{ds_I}{ds_c}$, or in another word $\frac{ds_c}{ds_I} \neq 0$.

When $\frac{ds_c}{ds_I} = 0$, Eqn. (3.4) becomes $x_I \frac{dx_I}{ds_I} + y_I \frac{dy_I}{ds_I} = \frac{d(x_I^2 + y_I^2)}{ds_I} = 0$, which is a circular arc segment centered at the origin I in system F_I . The corresponding sections of tooth and cutter profiles are also circular arcs. There is no relative motion at the instant, and thus $\frac{d\psi}{ds_I} = \kappa_c \frac{ds_c}{ds_I} = 0$. Equations of cusp degenerate to $\frac{dx_I}{ds_I} = 0$ and $\frac{dy_I}{ds_I} = 0$, and a cusp point exists on the contact path. In this situation, undercutting can be eliminated by avoiding cusps on the contact path. The case of $\frac{ds_c}{ds_I} = 0$ indicates a non-undercutting situation for regular contact path curves containing no cusp (further discussion in section 4). When $\frac{ds_c}{ds_I} \neq 0$, s_c can be used to parameterize other variables in all expressions and calculations.

Being parameterized by s_c, the equation of conjugation can be expressed as

$$x_I + x_I \frac{dx_I}{ds_c} + y_I \frac{dy_I}{ds_c} = 0 \tag{3.7}$$

and the equations of cusp become

$$-y_I \kappa_c + \frac{dx_I}{ds_c} + 1 = 0 \tag{3.8}$$

$$x_I \kappa_c + \frac{dy_I}{ds_c} = 0 \tag{3.9}$$

The two equations of cusp, (3.8) and (3.9), are usually correlated. Hence only one of them is needed to express the undercutting-free condition. If Eqn. (3.8) holds, then multiplying Eqn. (3.8) by x_I and subtracting Eqn. (3.7) from it yields $y_I\left(x_I \kappa_c + \frac{dy_I}{ds_c}\right) = 0$, which reveals that Eqn. (3.9) holds when $y_I \neq 0$. Similarly, multiplying Eqn (3.9) by y_I and subtracting Eqn. (3.7) from it yields $x_I\left(-y_I \kappa_c + \frac{dx_I}{ds_c} + 1\right) = 0$, which reveals that Eqn. (3.8) holds when $x_I \neq 0$. In other words, if $y_I \neq 0$ and $x_I \neq 0$, either Eqn. (3.8) or Eqn. (3.9) is adequate to express the equation of cusp. .

The two equations of cusp are the same at the origin of F_I too. Assume contact point C is denoted by a polar form of $\overrightarrow{IC} = \lambda\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ with respect to F_I where θ is the pressure angle s.t. $\sin\theta\cos\theta \neq 0$. The Eqn. (3.7) can be rewritten as

$$1 + \frac{dx_I}{ds_c} + \tan\theta \frac{dy_I}{ds_c} = 0 \tag{3.10}$$

in terms of pressure angle q. Checking equations of cusp Eqn. (3.8) and Eqn. (3.9) at the origin, there are $\frac{dx_I}{ds_c} = -1$ and $\frac{dy_I}{ds_c} = 0$, which are equivalent to each other under Eqn. (3.10). The above discussion is summarized below.

In system F_I, equations of cusp Eqn. (3.8) and Eqn. (3.9) are equivalent when x_I ¹ 0 and y_I ¹ 0, or when $x_I = 0$ and $y_I = 0$. The following theorem can thus be concluded.

Theorem 1: For any contact path, if either Eqn. (3.8) or Eqn. (3.9) does not hold, the corresponding conjugate curves are undercutting-free.

The theorem gives the sufficient condition for non-undercutting. The following sections discuss the use of parametric curves in conjugation design.

4 PARAMETRIC CONTACT PATH

Parametric curves and surfaces are widely used in CAD/CAM. Spline curves are special parametric curves. Spline curves can offer flexibility and freedom to the design of conjugate curves to enhance performance and contact quality.

A conjugate curve on a body may contain several sections. A very common and important application of conjugation design is in gearing. On a gear, the tooth profile usually has two sections (Fig. 3). In system F_I, the two sections correspond to a region in quadrants 1 and 3, i.e. $xy > 0$ and the origin, and the other region in quadrants 2 and 4, i.e. $xy < 0$ and the origin (Fig. 3). The straight segment $x_I = 0$ or $y_I = 0$ is excluded from the conjugation design due to practicality concern. In another word, pressure angle satisfies $\sin\theta\cos\theta \neq 0$. To avoid undercutting, a parametric contact path in F_I cannot have a cusp unless the cusp is at the end of the path. Cusp or undercutting will be detected in an open region where the definition of differentiation is regular.

The contact path (or curve) in F_I can be expressed in parametric form by $\vec{r}_I = \begin{pmatrix} x_I(u) \\ y_I(u) \end{pmatrix}$, where $u \in [u_0, u_1]$. Cusp detection is carried out in an open interval of $u \in (u_0, u_1)$.

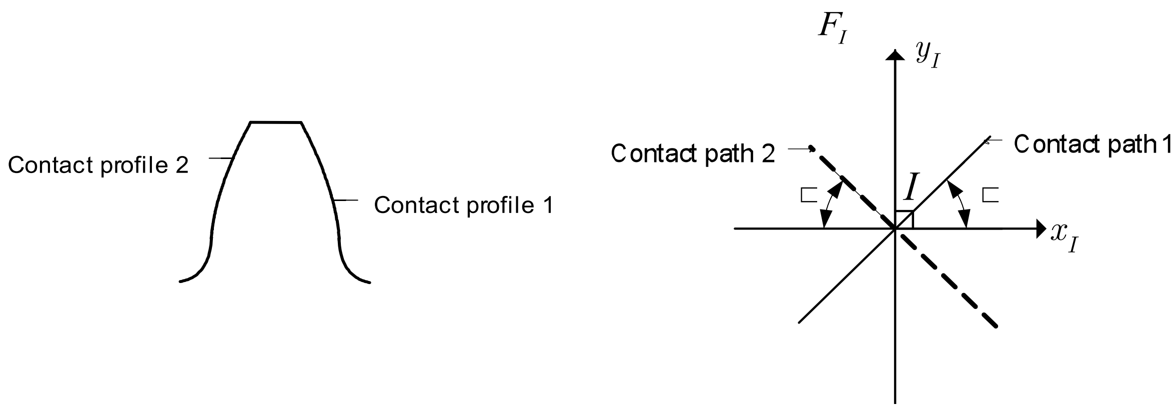


Fig. 3: Two sides of an involute tooth profile and their corresponding contact paths.

Let $\vec{r}_I$ be a regular curve, i.e. there is no u such that $\frac{d\vec{r}_I}{du} = 0$ or $\frac{ds_I}{du} = 0$ $u \in (u_0, u_1)$. Under this assumption, the special situation of $s_c^{'} = \frac{ds_c}{du} = 0$ (i.e. $\frac{ds_c}{ds_I} = 0$) will not cause undercutting according to the discussion in the previous section. This case can be integrated into the general parametric curve design or considered as a trivial circular section. From the standpoint of conjugation, $s_c^{'} = 0$ means that there are more than one contact point on the contact profile or contact path at the same time. To simplify design validation, one can check sections with $\frac{ds_c}{du} \neq 0$, s.t. $s_c^{'} \neq 0$, which are one-to-one correspondence contact sections. The pattern of more than one contact point can always be resolved by using multiple connected one-to-one contact sections.

$s_c^{'} > 0$ and $s_c^{'} < 0$ correspond to different directions of contact pattern. If contact position on curve $\vec{r}_I$ is continuous in the ascending direction, s.t. $\frac{ds_c}{ds_I} > 0$ (s_I is the length parameter of $\vec{r}_I$), then $\frac{ds_c}{ds_I} = \frac{1}{\sqrt{x_I^{'2} + y_I^{'2}}}\frac{ds_c}{du} > 0$, and thus $s_c^{'} = \frac{ds_c}{du} > 0$. On the other hand, $\frac{ds_c}{ds_I} < 0$ leads to $s_c^{'} < 0$.

In system F_I , the equation of conjugation, Eqn. (3.7), becomes

$$x_I s_c^{'} = - \left(x_I^{'} x_I + y_I^{'} y_I \right) \tag{4.1}$$

Equations (3.8) and (3.9) are equivalent. If Eqn. (3.9) is used, the equation of cusp becomes

$$x_I \kappa_c s_c^{'} = -y_I^{'} \tag{4.2}$$

The above two equations are the equation of conjugation and equation of cusp with a parametric contact path representation.

5 B-SPLINE AND GEAR TOOTH PROFILES

In planar gearing, a straight moving centrode can be viewed as a virtual rack-cutter and the fixed centrode represents one of the mated gears (Fig. 1). Thus, a gearing system includes two mated gears with two gear centrodes and one rack-cutter with a straight-line centrode. As the three centrodes roll against each other, the tooth profiles on the gears and rack-cutter roll and slide along a common contact path. In other words, these three profiles are conjugate to one another. Hence, the equation of conjugation between the conjugate curves on any two of the three moving bodies is the same.

Let the gear transmission ratio be $i_{12} = -\dfrac{d\phi_1}{d\phi_2}$, where ϕ_1 and ϕ_2 are the rotation angles of the mated gears (counterclockwise is assumed as the positive direction). In the relative motion with respect to the rack-cutter, each gear has its own equation of cusp. Hence, there are two equations for undercutting condition. The two rotation angles follow a predesigned function of time t by the defined angular velocity. Assume $\dfrac{dt}{d\phi_1} > 0$ and without loss of generality, let $\dfrac{dt}{d\phi_1} = 1$ or $t = \phi_1$ hold constantly. For gear 1, $\dfrac{ds_c}{dt} = R_1\left(t\right)$ and $R_1 = \dfrac{1}{\kappa_{c1}}$ is the instantaneous radius of curvature of the centrode in gear 1, which cannot be zero in practice. The rolling distance of the centrode is $s_c = \displaystyle\int_{t_0}^{t} R_1\left(x\right)dx$. Furthermore, t can be parameterized by u s.t. $t^{'} = \dfrac{dt}{du} = \dfrac{dt}{ds_c}\dfrac{ds_c}{du} = \dfrac{1}{R_1\left(t\right)} s_c^{'} = \kappa_{c1} s_c^{'}$.

For most practices in gearing, contact gear teeth are chosen in a domain restricted between the equations of cusp. For example, $x_I t^{'}$ should be between $y_I^{'}$ and $-i_{12} y_I^{'}$ for contact path curve. For gears in Fig. 1, assume $t^{'} > 0$ for one-to-one correspondence contact, and then $y_I^{'} < 0$, $y_I^{'} \le x_I t^{'} \le -i_{12} y_I^{'}$.

The specific equation of conjugation in F_I is

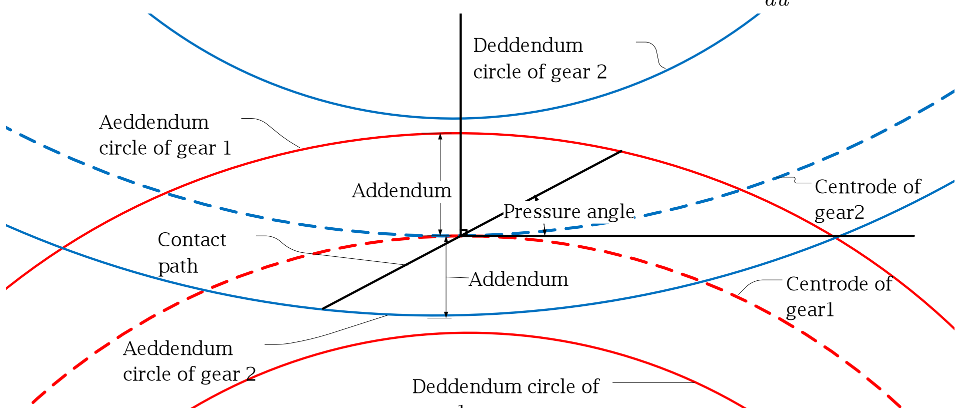
$$x_I t^{'} = \frac{1}{R_1\left(t\right)} \left(-x_I^{'} x_I - y_I^{'} y_I \right) \tag{5.1}$$

There are two selected equations of cusp for contact path, $x_I t^{'} \ne y_I^{'}$ and $x_I t^{'} \ne -i_{12} y_I^{'}$ corresponding to gear 1 and gear 2 respectively. To avoid undercutting, $x_I t^{'}$ must stay away from $y_I^{'}$ and $-i_{12} y_I^{'}$.

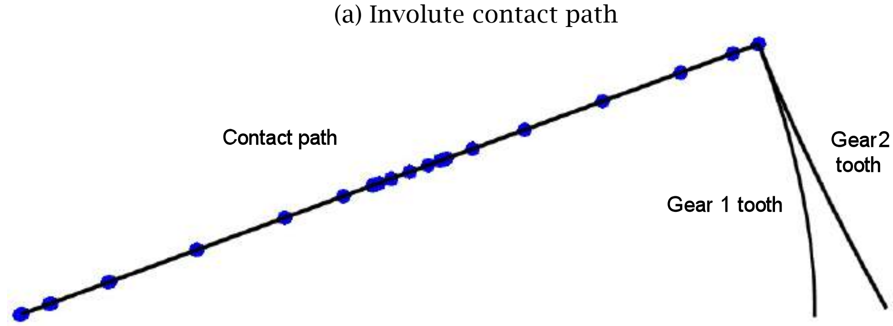
The general conditions for a valid contact curve are related to parameter u . To test these conditions and hence the validity of the synthesized contact curves, one may search for the extreme values whenever undercutting occurs. Due to the complexity of the undercutting conditions, a nonlinear method, such as Newton-Raphson [13], is necessary for finding the extreme values. The existence of undercutting must be checked whenever the designed curve changes.

Generally, nonlinear optimization is difficult to implement in conjugation design. Even with a simple cusp condition, the convexity of optimization about u is not guaranteed. Furthermore, nonlinear optimization is time consuming for interactive tooth profile design even for simple B-splines. The discussion below offers a simple solution for validation of undercutting.

Since the tangent of a B-spline is a lower order B-spline, the boundaries of the tangent of a designed B-spline contact path can be found. These conditions are possible due to the control polygon property of B-splines—a B-spine curve is inside the convex hull of its control points. Given knot vector $\{u_i\}_{i=0}^{i=n+p+1}$, a B-spline curve can be represented by $\vec{r}(u) = \sum_{i=0}^{n} P_i N_{i,p}(u)$, $u \in [0,1]$, where P_i are control points and $N_{i,p}$ are basis functions of order p [11]. The B-splines used in this paper interpolate end control points, with $u_0 = ... = u_p = 0$ and $u_{n+1} = ... = u_{n+p+1} = 0$. The tangent function is $\frac{d\vec{r}(u)}{du} = \sum_{i=0}^{n-1} Q_i N_{i+1,p-1}(u)$, where $Q_i = \frac{p}{u_{i+p+1} - u_{i+1}}(P_{i+1} - P_i)$ is a tangent control point. If the boundaries of $\frac{d\vec{r}(u)}{du}$ are found, then keeping Q_i inside the boundaries can guarantee that $\frac{d\vec{r}(u)}{du}$ is bounded properly.



(a) Involute contact path



(b) B-spline contact path

Fig. 4: Involute contact path: (a) Contact path between the addendum circles of gears 1 and 2. (b) B-spline contact path and the corresponding gear tooth profiles. Gear teeth are in initial contact position. The straight segments of contact path lead to involute profiles.

The expressions of tangent control points Q_i are one order lower than that of the B-spline curve and lead to a linear function of control points. It is very efficient to test the difference functions of control points. If there are n control points, then there are n - 1 tangent control points to check. The rest of this section is focused on how to find the boundaries of a tangent function.

For a one-to-one correspondence contact pattern, time parameter t satisfies either $t' > 0$ or $t' < 0$, depending on the rotation direction of gears and the parameterization directions of the tooth profiles. Without loss of generality, $t' > 0$ is assumed and only one side of a tooth is considered. The positions of gears are shown in Fig. 1. The section with $y_I' \le x_I t' \le -i_{12} y_I'$ is considered for a regular design. The considered curves here are the contact path and the tooth curves on gear 1 and gear 2. To use B-splines in design, control points are added to the contact path (Fig. 4).

Simple results can be obtained for a B-spline contact path in F_I if manipulated properly. Let the boundaries of x_I and y_I be the same as those of involute gears s.t. $x_{inf} \le x_I \le x_{sup}$, $y_{inf} \le y_I \le y_{sup}$. To make $y_I' \le -i_{12} y_I'$ reasonable, another constraint $y_I' < 0$ is added. The following discusses how to bound derivatives x_I' and y_I'.

Let y_I be uniform about parameter u such that y_I' is a negative constant. This is possible with B-splines if y direction is uniformly parameterized. When $0 \le x_I \le x_{sup}$, then $0 \le x_I t' \le -i_{12} y_I'$, which yields, from Eqn. (5.1), $-\dfrac{y_I'\left(y_{sup} - i_{12} R_1\right)}{x_{sup}} \le x_I' \le 0$. Likewise, when $x_{inf} \le x_I \le 0$, then $y_I' \le x_I t' \le 0$, which yields, from Eqn. (5.1), $-\dfrac{y_I'\left(y_{inf} + R_1\right)}{x_{inf}} \le x_I' \le 0$. Notice that $y_{sup} < i_{12} R_1$ and $y_{inf} >$ - R_1 because a gear tooth curve cannot reach gear rotation center. The above results can be summarized as

$$\max\left(-\frac{y_I'\left(y_{sup} - i_{12} R_1\right)}{x_{sup}}, -\frac{y_I'\left(y_{inf} + R_1\right)}{x_{inf}}\right) \le x_I' \le 0 \tag{5.2}$$

Similarly, let x_I' be a constant that could be positive or negative. In the first quadrant of F_I, i.e. $0 \le x_I \le x_{sup}$ and $0 \le y \le y_{sup}$, there is $0 \le x_I t' \le -i_{12} y_I'$, which yields, from Eqn. (5.1), $y_I' \le \min\left\{-\dfrac{x_I' x_{sup}}{\left(y_{sup} - i_{12} R_1\right)}, -\dfrac{x_I' x_{sup}}{y_{sup}}\right\}$. Likewise, in the third quadrant of F_I, i.e. $x_{inf} \le x_I \le 0$, $y_{inf} \le y_I \le 0$, there is $y_I' \le x_I t' \le 0$, which yields, from Eqn. (5.1), $y_I' \le \min\left\{-\dfrac{x_I' x_{inf}}{\left(y_{inf} + R_1\right)}, -\dfrac{x_I' x_{inf}}{y_{inf}}\right\}$. In summary,

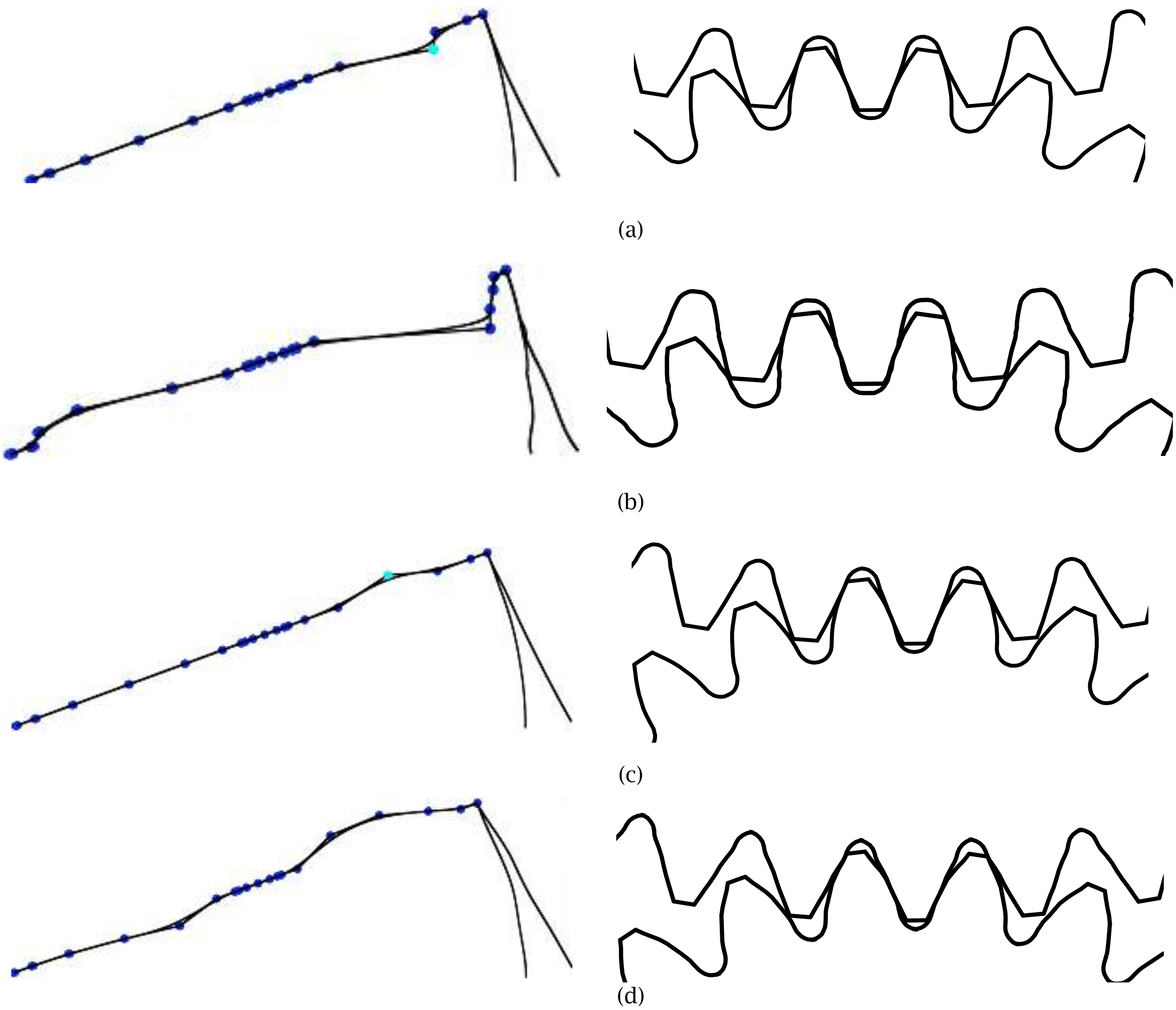
$$\begin{cases} y_I' \le \min\left(-\dfrac{x_I' x_{sup}}{y_{sup} - i_{12} R_1}, -\dfrac{x_I' x_{inf}}{y_{inf} + R_1}\right), & x_I' \le 0 \\ y_I' \le \min\left(-\dfrac{x_I' x_{sup}}{y_{sup}}, -\dfrac{x_I' x_{inf}}{y_{inf}}\right), & x_I' > 0 \end{cases} \tag{5.3}$$

Once the boundaries of the derivatives x_I' and y_I' are derived, they can be used to restrict tangent control points of B-spline contact path to the discussion above.

6 EXAMPLES

Gear tooth profile design and modification with B-spline contact path is illustrated in the following four examples. Initially involute profiles will be used and the gear parameters and contact path can be selected or determined in the conventional way [8]. B-splines are then used to represent and modify contact path.

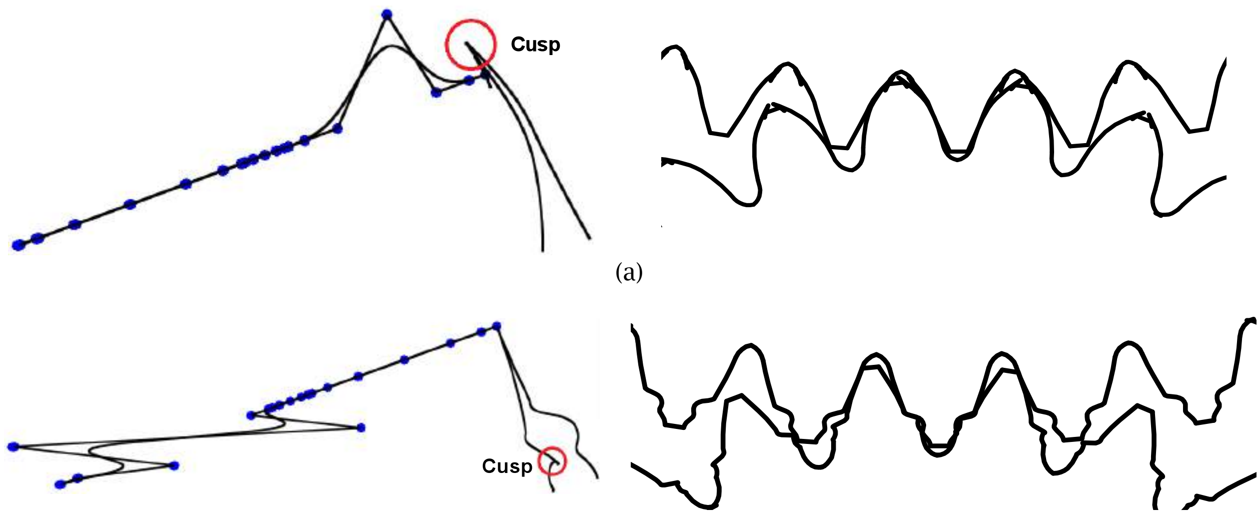
With involute profiles, the contact path is straight (Fig. 4). The following gear parameters are selected: diametral pitch $P_d = 0.5$, pressure angle 20°, addendum $add = \frac{1}{P_d}$, deddendum $dedd = \frac{1.25}{P_d}$, gear ratio $i_{12} = 75/25$.



(a) (b) (c) (d)

Fig 5: Control points and contact paths of four feasible designs.

To obtain a uniform parameterization, one can interpolate a straight line segment with a one-degree B-spline first. Then elevate the degree to what is needed (three is used here). Insert all the knots. The resulted B-spline should be uniformly parameterized. Once the contact path is given, Eqn. (5.1) is solved for time $t(u)$, and then tooth curves can be derived from kinematic relations between moving centrodes [8]. Since deddendum is greater than addendum, the gap between gear contact area and deddendum circle is kept the same as the initial involute curve. Only contact sections are modified.



(a)

(b)

Fig. 6: Control points and contact path leading to undercutting. (a) Cusps are at the top of gear teeth. (b) Cusps are in the middle of the tooth curves of gear 1.

Each contact path has three sections. One lies in quadrant 1; one small section passes through the origin; the third section lies in quadrant 3. These three sections form a general gear tooth contact path. The origin works as a bridge connecting sections in quadrants 1 and 3 to form a complete design domain. In the following control point changing process, section 2 of the contact path will remain a short straight segment so that the idea of a single quadrant contact path design can be seen clearly from these general examples.

The left figures of Fig. 5 and Fig. 6 show contact path control polygon, contact path curve, and the corresponding tooth profiles on gears 1 and 2 (see Fig. 1 and Fig. 4 for detail). The right figures are the corresponding gears. Gear 1 is placed under gear 2 in each figure. Gear tooth profile design is changed by manipulating control points of the contact path.

Four sets of feasible tooth profiles with the corresponding contact paths are shown in Fig. 5. Figures (5a) and (5b) satisfy the inequality Eqn. (5.2) with a constant $y_I^{'} < 0$ while figures (5c) and (5d) satisfy the inequality Eqn. (5.3) with $t^{'} > 0$ and a constant $x_I^{'} < 0$. All gears contain non-involute tooth profiles. One may easily observe that control points of the contact path can be moved freely within the design constraints. Hence the proposed method offers a free-form approach that is ideal for synthesizing tooth profiles with the optimal contact properties.

Fig 6 shows two instances with undercutting. These gears cause problems such as interference and tip contact.

7 CONCLUSION

The paper extends the application of splines from single curve design to the design of conjugate curves. The equation of cusp for undercutting avoidance and general conditions relating a parametric contact path to a pair of conjugate curves are presented. Corresponding to a selected contact path, a pair of conjugate curves can be identified. It demonstrates how splines can be used to manipulate contact path leading to the desired conjugate curves. The method offers unlimited flexibility and freedom to the design of conjugate surfaces as splines to single surfaces. The use of B-splines for conjugation design is demonstrated in the examples, in which the initial contact path comes from involute conjugation. Other initial options directly associated to design parameters such as diametral pitch, contact ratios, etc. may also be used.

In gearing, the proposed free-form conjugation could essentially change the practice of conjugate tooth profile design. The proposed method includes the traditional involutes. It effectively liberates the tooth profile design. Conjugate curves are no longer limited to few traditional curves with specific conjugate geometric characteristics. Desired conjugate curves can be synthesized and manipulated from an unlimited pool of splines for the best performance. This new design freedom is especially important for improving the power density in large gears with heavy machinery—size reduced for the same work load.

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