Computer-Aided Design and Applications
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Optimal Sampling of Parametric Surfaces

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ABSTRACT

We study the problem of optimally sampling parametric surfaces by means of reparameterization. A criterion is first formulated for measuring the parameterization quality of a given surface. According to this criterion, the optimal parameterization is identified for the surface by exploring admissible reparameterizations. Then the optimal sampling of the surface is obtained by uniformly sampling the parameter domain of the optimal parameterization.

Keywords: parametric surface, optimal sampling, surface reparameterization.
DOI: 10.3722/cadaps.2012.55-60

1 INTRODUCTION

Parametric surfaces are widely used to specify surfaces in CAD applications. We consider the problem of optimal sampling of a rectangular surface patch, which is defined as follows: Given a parametric surface patch defined over the domain $[0,1]\times[0,1]$, how to compute a rectangular grid of the parameter domain so as to decompose the surface into quad patches such that 1) each quad patch should have their sides with equal length; and 2) the shapes and sizes of all the quad patches should be the same as much as possible. Many algorithms for surface rendering, intersection, and tessellation applications depend greatly on the sampling of the surfaces as properly sampled surfaces can improve the performances of these algorithms.

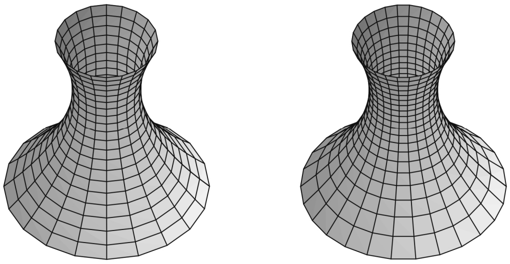

Fig. 1: Two Samplings of a surface of revolution: (Left) A non-uniform sampling of the surface is given by uniformly sampling the parameter domain; (Right) uniform sampling of the surface is given by a non-uniform sampling of the parameter domain.

It is clear that, in general, simply performing a uniform sampling of the parameter domain would not yield a uniform sampling of the surface itself, because the mapping from the domain to the surface can be highly distortive. For example, consider a parametric surface of revolution in Fig. 1: uniformly sampling the parameter domain leads to a non-uniform sampling of the surface (see Fig. 1(left)). In order to get a uniform sampling of this surface, where the sampled parameter curves segment the surface into squared regions as shown Fig. 1(right), we may have to somehow non-uniformly sample the parameter domain.

As a related problem, uniform sampling of a curve has extensively been studied [1-8]. There, the problem is how to compute the optimal parameterization of a parametric curve that is as close as possible to the arc length parameterization, since the arc length parameterization produces a uniform point sampling for constant sampling parameter intervals; this is also called the *unit-speed property.* However, there has been little research on uniform sampling of parametric surfaces. It is proposed in [9] to compute the optimal parameterization for Bézier surfaces using linear Mobius transformations and quadratic transformations as reparameterization functions. But they consider only minimizing the energy of boundary curves without regard to the sampling quality in the interior of the surface.

We present a solution to the problem of computing the optimal sampling for parametric surfaces. We first propose a novel optimality criterion for the parameterization of a surface, i.e. measuring its quality using deviation from an ideal uniform parameterization. Then we search for the optimal parameterization by exploring admissible parameter transformations to minimize this deviation. Finally, an optimal sampling of the surface is obtained by uniformly sampling the new parameter domain of the optimal parameterization.

2 OPTIMALITY CRITERION FOR SURFACE PARAMETERIZATIONS

In [6], Farouki introduces optimal parameterizations and proposes an optimality criterion for parametric curve to measure the deviation of its parameterization from arc length parameterization. For a parametric curve $r(t)$ of unit arc length and over the interval $[0,1]$, clearly, $\|r'(t)\|$ describes the parametric speed of the curve, i.e., the change rate of its arc length with respect to the parameter t. Then the optimality is measured by:

$$I = \int_0^1 (\|r'(t)\| - 1)^2 dt \tag{2.1}$$

Or equivalently:

$$J = \int_0^1 \|r'(t)\|^2 dt \tag{2.2}$$

It is evident that I is nonnegative, and the ideal parameterization of the curve $r(t)$ is the arc length parameterization with uniform speed $\|r'(t)\| = 1$, which makes $I = 0$.

For the optimal sampling of a surface, the requirement is more involved. The requirement of the optimal surface sampling problem amounts to finding a parameterization of the surface such that the difference between the magnitudes of its two partial derivatives is as small as possible everywhere.. It follows that, for a parametric surface $P(u,v)$ over a unit square parameter domain $u, v \in [0,1]^2$, the optimality criterion for measuring its parameterization is:

$$H = \int_0^1 \int_0^1 \left(\left\| \frac{\partial P(u,v)}{\partial u} \right\|_2^2 - \left\| \frac{\partial P(u,v)}{\partial v} \right\|_2^2 \right)^2 du dv \tag{2.3}$$

which measures the deviation of $P(u,v)$ from an ideal uniform parameterization. Here, $\left\| \frac{\partial P(u,v)}{\partial u} \right\|_2$ and $\left\| \frac{\partial P(u,v)}{\partial v} \right\|_2$ are the parametric speeds of the surface with respect to the two parameters u and v, respectively. For a parametric surface with an ideal uniform parameterization, these two parametric speeds are equal everywhere, i.e. $H = 0$, otherwise, $H > 0$. Intuitively, for surfaces with $H = 0$, any small square in the parameter domain is mapped to a curved boundary rhombus on the surface, and uniformly sampled square grids in the parameter domain would produce a uniform sampling of the surface, e.g. Fig. 1(b).

3 PARAMETERIZATION OPTIMIZATION

In general, the given parameterization of a parametric surface is not in the form of uniform parameterization ($H = 0$). In this section, we shall propose an algorithm for finding the optimal parameterization for a given parametric surface. The main idea is to first introduce a class of parameter transformations, and then minimize Eqn. (2.3) over all transformation variables to identify the optimal composite reparameterization that is as close as possible to an ideal uniform parameterization.

We will limit our discussion to Bézier surfaces, but the basic idea applies to other parametric surfaces. Consider a Bézier surface:

$$P(u,v) = \sum_{i=0}^{m}\sum_{j=0}^{n} B_i^m(u)B_j^n(v)\,K_{i,j} \qquad u \in [0,1], v \in [0,1] \tag{3.1}$$

$K_{i,j}$ are control points, $B_i^m(u) = \binom{m}{i}u^i(1-u)^{m-i}$ and $B_j^n(v) = \binom{n}{j}v^j(1-v)^{n-j}$ are Bernstein polynomials. A surface sampling refers to a set of sampled parameter lines on the surface defined with respect to discrete, fixed parameter increments. A parameter line of $P(u,v)$ for a constant value $u = u_0$ takes the form $L_{u_0}(v) = P(u,v)|_{u=u_0}$, which is a Bézier curve of degree n with $B_i^m(u_0)K_{i,j}$ as its control points. Likewise, the parameter line for a constant value $v = v_0$ is $L_{v_0}(u) = P(u,v)|_{v=v_0}$, which is of degree m with $B_j^n(v_0)K_{i,j}$ as its control points.

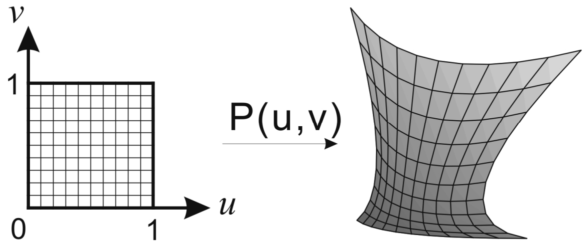

Fig. 2: Sampling of a Bézier surface: uniformly sampling the parameter domain (left) leads to non-uniform sampling of the surface (right).

Consider the Bézier surface in Fig. 2, for example. A uniform sampling of its parameter domain $u, v \in [0,1]^2$ on the left produces non-uniform sampling of the surface on the right. This indicates that the given polynomial parameterization $P(u,v)$ of a Bézier surface is not the ideal uniform parameterization under the optimality criterion in Eqn. (2.3).

Given a Bézier surface $P(u,v)$ of the form Eqn. (3.1), each parameter is subject to a transformation: $u = u(s)$ and $v = v(t)$, called *reparameterization*. Substituting these two transformations in Eqn. (3.1) yields

$$Q(s,t) = P(u,v)|_{u=u(s),v=v(t)} = \sum_{i=0}^{m}\sum_{j=0}^{n} B_i^m(u(s))B_j^n(v(t))\,K_{i,j} \tag{3.2}$$

Finding optimal parameterization of the surface amounts to minimizing the following objective function in Eqn. (2.3) over free variables of the reparameterizations $u = u(s)$ and $v = v(t)$.

$$H = \iint \left(\left\| \frac{\partial Q(s,t)}{\partial s} \right\|_2^2 - \left\| \frac{\partial Q(s,t)}{\partial t} \right\|_2^2 \right)^2 ds dt \tag{3.3}$$

Now suppose that the parameter transformations $u = u(s)$ and $v = v(t)$ are subject to the constraints:

$$u(0) = 0,\ u(1) = 1,\ u'(s) \geq 0\ \ s \in [0,1] \quad v(0) = 0,\ v(\alpha) = 1,\ v'(t) \geq 0\ \ t \in [0,\alpha] \tag{3.4}$$

These constraints mean that $u(s)$ and $v(t)$ are monotonically increasing bijective mappings. The number α represents the aspect ratio of the new parameter domain. For the ideal uniform parameterization ($H = 0$) of a surface, α is approximately equal to the aspect ratio of the surface due to the uniform property of the parameterization, i.e. small squares in the parameter domain are mapped to rhombus-like regions on the surface (See Fig. 3.). That means α depends on the shape of the surface, and therefore should be treated as a variable in optimization.

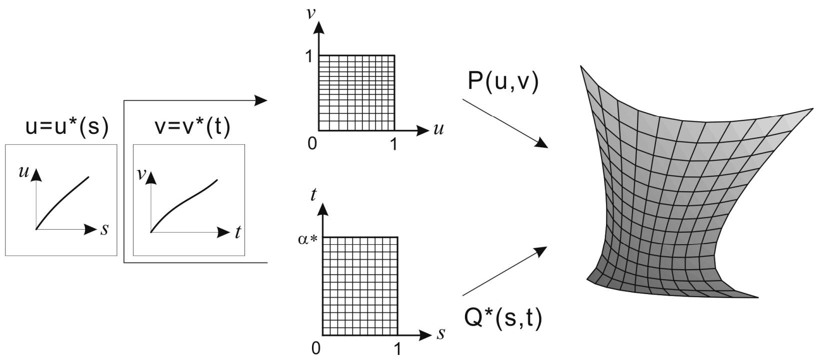

Fig. 3: Optimal sampling of the input Bézier surface in Fig. 2: After the determination of the optimal parameterization $Q^*(s,t)$, an optimal sampling of the surface (right) reduces to uniformly sampling the parameter domain of $Q^*(s,t)$ (middle bottom).

Combining Eqn. (3.2) (3.3) and (3.4) yields

$$H = \int_0^\alpha \int_0^1 \left(\left\| \sum_{i=0}^m \sum_{j=0}^n B_j^n(v(t))\, K_{i,j} \frac{d}{ds}(B_i^m(u(s))) \right\|_2^2 - \left\| \sum_{i=0}^m \sum_{j=0}^n B_i^m(u(s))\, K_{i,j} \frac{d}{dt}(B_j^n(v(t))) \right\|_2^2 \right)^2 dsdt \quad (3.5)$$

$$= \int_0^\alpha \int_0^1 \left(\left\| \sum_{i=0}^m \sum_{j=0}^n B_j^n(v(t))\, K_{i,j} \frac{d}{du}(B_i^m(u)) \frac{d}{ds}(u(s)) \right\|_2^2 - \left\| \sum_{i=0}^m \sum_{j=0}^n B_i^m(u(s))\, K_{i,j} \frac{d}{dv}(B_j^n(v)) \frac{d}{dt}(v(t)) \right\|_2^2 \right)^2 dsdt$$

(3.6)

Substituting $t = \alpha\bar{t}$, we have

$$H = \alpha \int_0^1 \int_0^1 \left(\left\| \sum_{i=0}^m \sum_{j=0}^n B_j^n(v(\alpha\bar{t}))\, K_{i,j} \frac{d}{du}(B_i^m(u)) \frac{d}{ds}(u(s)) \right\|_2^2 \right.$$
$$\left. - \left\| \frac{1}{\alpha} \sum_{i=0}^m \sum_{j=0}^n B_i^m(u(s))\, K_{i,j} \frac{d}{dv}(B_j^n(v)) \frac{d}{d\bar{t}}(v(\alpha\bar{t})) \right\|_2^2 \right)^2 dsd\bar{t} \quad (3.7)$$

Let $u(s)$ and $v(t)$ be confined to be polynomials of degree d_u and d_v for the ease of manipulation:

$$u(s) = \sum_{p=0}^{d_u} U_p s^p, \; v(t) = \sum_{p=0}^{d_v} V_p t^p \quad (3.8)$$

Substituting Eqn. (3.8) in Eqn. (3.7) yields:

$$H(X) = \alpha \int_0^1 \int_0^1 \left(\left\| \sum_{i=0}^m \sum_{j=0}^n B_j^n\left(\sum_{p=0}^{d_v} V_p \alpha^p \bar{t}^p \right) K_{i,j} \frac{d}{du}(B_i^m(u)) \frac{d}{ds}\left(\sum_{p=0}^{d_u} U_p s^p\right) \right\|_2^2 \right.$$
$$\left. - \left\| \frac{1}{\alpha} \sum_{i=0}^m \sum_{j=0}^n B_i^m\left(\sum_{p=0}^{d_u} U_p s^p\right) K_{i,j} \frac{d}{dv}(B_j^n(v)) \frac{d}{d\bar{t}}\left(\sum_{p=0}^{d_v} V_p \alpha^p \bar{t}^p\right) \right\|_2^2 \right)^2 dsd\bar{t} \quad (3.9)$$

Note that after integration $H(X)$ is simply a polynomial over the variable set $X = \{U_0 \dots U_{d_u}, V_0 \dots V_{d_v}, \alpha\}$, i.e. polynomial coefficients and aspect ratio of the parameter domain.

In summary, the determination of the optimal parameterization amounts to minimizing $H(X)$ in Eqn. (3.9) over the variable set X and subject to the constraints in Eqn. (3.4). Denote $X^* = argmin\, H(X)$. Then the parameterization becomes:

$$Q^*(s,t) = Q(s,t)|_{X=X^*} \qquad s \in [0,1],\; t \in [0,\alpha^*] \quad (3.10)$$

$Q^*(s,t)$ is the parameterization that is as uniform as possible, i.e. the optimal parameterization of the surface, which exhibits optimal samplings: an optimal sampling of the surface can now be obtained by uniformly sampling the new parameter domain $[0,1] \times [0,\alpha^*]$ (see Fig. 3.).

4 ALGORITHM AND EXPERIMENTS

We now present the algorithm for computing optimal sampling of a parametric surface, which is assumed to be a Bézier surface of degree $m \times n$ in the form of Eqn. (3.1):

1. Represent the parameter transformations $u(s)$ and $v(t)$ as polynomials of degree $d_u = m+1$ and $d_v = n+1$, respectively (see Eqn. (3.8)). Compute the reparameterized surface representation $Q(s,t)$ (see Eqn. (3.2)) by combining $u(s)$ and $v(t)$.

2. Formulate the deviation of $Q(s,t)$ from ideal uniform parameterization, based on the criterion H on $Q(s,t)$ (see Eqn. (3.3));

3. Minimize the objective function $H(X)$ in Eqn. (3.9) subject to the constraints in Eqn. (3.4). The Sequential Quadratic Programming (SQP) solver is used in our experiments to solve this constrained Non-Linear Programming (NLP) problem. The minimum $X^* = argmin\, H(X)$ gives the optimal parameterization $Q^*(s,t)$ (see Eqn.(3.10)).

5. Uniformly sampling the parameter domain of $Q^*(s,t)$ to produce an optimal sampling of the given surface.

To give an example, we take the Bézier surface $P(u,v)$ shown in Fig. 2 as input and perform the above algorithm (see Fig. 3). In this example, cubic polynomials are adopted as parameter transformations, since the input Bézier surface is of degree $m = n = 2$. The initial value $\alpha = 1.2901$ is used, which is estimated by the approximate aspect ratio of the surface

$$\alpha = \frac{\int_0^1 \int_0^1 \left\| \frac{\partial P(u,v)}{\partial u} \right\|_2 du dv}{\int_0^1 \int_0^1 \left\| \frac{\partial P(u,v)}{\partial v} \right\|_2 du dv} \quad (3.11)$$

The optimized value α^* is 1.2771. The optimized parameter transformations $u^*(s)$ and $v^*(t)$ (see Fig. 3 (left)) transform the (*s, t*) domain into the (*u, v*) domain. As shown in Fig. 3, a uniform sampling of the (*s, t*) domain $[0,1] \times [0,\alpha^*]$ produces an optimal sampling of the surface, whose corresponding sampling in the original (*u, v*) domain is shown in Fig. 3(middle top). More examples of optimal surface sampling produced with our method are given in Fig. 4 (in each example, left: input; right: output).

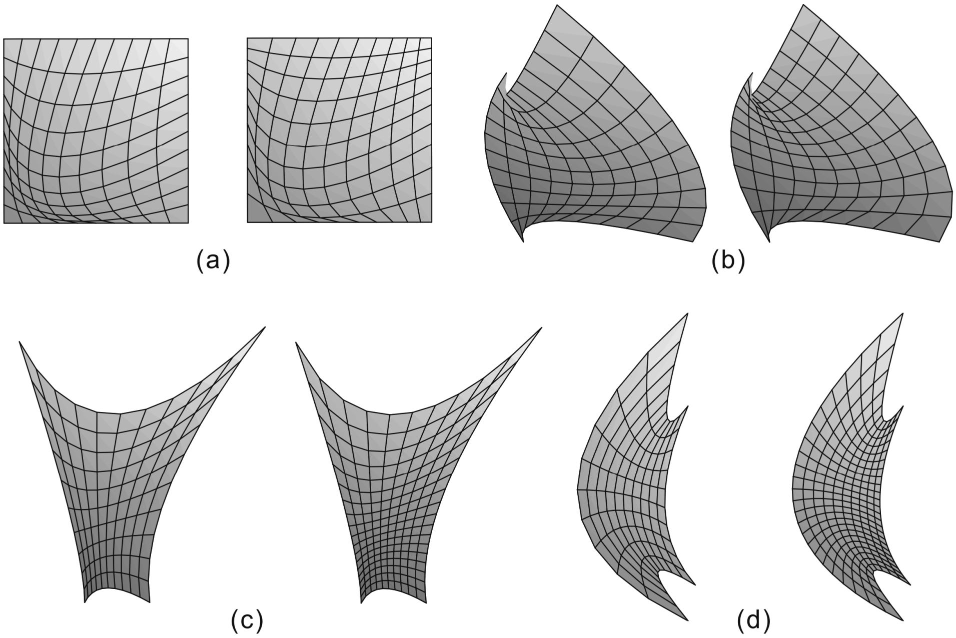

Fig. 4: Optimal sampling examples: in each example, left is the input and right is the output.

5 CONCLUSION

In this paper, we addressed the problem of computing the optimal sampling of parametric surfaces. We first propose an optimality criterion for measuring the parameterization quality of a given surface. According to this criterion, we search for the optimal reparameterization by exploring admissible parameter transformations. Finally, the determination of optimal sampling for the surface reduces to uniformly sampling the parameter domain of the identified optimal parameterization. The efficacy of the proposed method is demonstrated with several examples.

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