# Improved Derivative Bound Estimations of Rational Bézier Curves

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**Abstract.** The estimation of bounds on derivatives of rational Bézier curves has significant application in Computer Aided Geometric Design (CAGD). This paper proposes a piecewise method to estimate the derivative bounds of the rational Bézier curves. This method applies the Bernstein basis functions' extreme value characteristics and fundamental inequalities to estimate the derivative bounds of the rational Bézier curves. Both theoretical analysis and numerical examples show that our bound is sharper than existing ones. The proposed method has an excellent convergence effect for smooth rational Bézier curves with severe weights and control points.

**Keywords:** Rational Bézier curves; Derivative bound; Bernstein basis functions; Estimation.
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## 1 INTRODUCTION

Bézier curves and surfaces are widely used in free-form geometry design, and rational Bézier curves can accurately represent certain analytical curves and surfaces. Significantly, the estimation of derivative bounds is one of the few vital global properties of curves, which has essential applications in computer-aided geometric design [1] and computer graphics [2].

In computer graphics and modeling, parametric curves and surfaces are often tessellated into piecewise linear segments for rendering [3], mesh generation [4,5], and surface intersection [6]. Moreover, various curve algorithms depend on the approximation error, such as linear approximation, creation of offset curves, and determination of inflection points or singularities. In such applications, the approximation error is typically taken as the maximal distance between the original and the approximating segments, determined by the global parameter interval or step size valid over the entire domain. One popular approach is establishing a relatively simple relationship between the step size and the bound of a rational curve or surface. The estimation of a bound should ideally be as accurate as possible to ensure a larger step size without violating the prescribed tolerance.

Many researchers have introduced estimation methods for the derivative bounds of rational Bézier curves [7-17,19]. A rational Bézier curve $R(t)$ of degree $n$ is given by the control points $P_i \in \boldsymbol{R}^3$ and corresponding positive weights $\omega_i \in \boldsymbol{R}^+$ as follows:

$$R(t) = \frac{\sum_{i=0}^{n} B_i^n(t)\omega_i P_i}{\sum_{i=0}^{n} B_i^n(t)\omega_i}, 0 \le t \le 1, \tag{1.1}$$

where are the Bernstein polynomials given by $B_i^n(t) = \begin{pmatrix} n \\ i \end{pmatrix} t^i (1-t)^{n-i}$ . In connection with this paper, $\left\| R'(t) \right\|$ means the 2-norm of the derivative of $R(t)$ .

With the present study, we want to propose a unified result about first-order derivative bounds of the rational Bézier curves based on the characteristics of Bernstein basis functions, the definition of derivatives, and some fundamental inequality relations. Moreover, the bounds given in this paper are sharper than the existing bounds, especially for smooth rational Bézier curves with abrupt changes in weights and control points, which have an excellent convergence effect.

## 2 PREVIOUS WORK

Many researchers have given the estimation method of the derivative bounds of the rational Bézier curves. Generally, a bound on the magnitude of the derivative of rational Bézier curves is the product of a positive real number and a module defined by control points and weights. Based on (1.1), Floater [7] presented two estimations for the rational Bézier curve as

$$\left\| R'(t) \right\| \le n \frac{W}{\omega} \max_{i,j} \left\| P_i - P_j \right\|, \tag{2.1}$$

$$\left\| R'(t) \right\| \le n \frac{W^2}{\omega^2} \max_{i} \left\| P_{i+1} - P_i \right\|, \tag{2.2}$$

where $W = \max_i \; w_i$ , $\omega = \min_i \; w_i$ .

Selimovic [9] obtained tighter bounds for rational Bézier curves as

$$\left\| R'(t) \right\| \le n \max_{i,j} \left\{ \max_i \frac{\omega_{i+1}}{\omega_i}, \max_i \frac{\omega_i}{\omega_{i+1}} \right\} \cdot \max_{i,j} \left\| P_i - P_j \right\|, \tag{2.3}$$

$$\left\| R'(t) \right\| \le n \max_{i,j} \left\{ \max_i \frac{\omega_{i+1}}{\omega_i}, \max_i \frac{\omega_i}{\omega_{i+1}} \right\}^n \cdot \max_i \left\| P_{i+1} - P_i \right\|. \tag{2.4}$$

Huang [10] presented such a bound as

$$\left\| R'(t) \right\| \le n \max_{i,j} \left\{ \frac{\left\| Q_{ij} \right\|}{\min \;\; w_i, w_{i+1}} \right\}, \tag{2.5}$$

where $Q_{ij} = \omega_{i+1}(P_{i+1} - P_i) - \omega_i (P_i - P_i)$ .

Deng [12] gave the following bound

$$\left\|R'(t)\right\| \le n\Omega(\frac{2M}{1+M})^{n-1}D\ , \tag{2.6}$$

where $\Omega = \max_i\left\{\frac{i}{n}\cdot\frac{\omega_{i-1}}{\omega_i}+\frac{n-i}{n},\frac{n-i}{n}\cdot\frac{\omega_{i+1}}{\omega_i}+\frac{i}{n}\right\}$, $M=\sqrt{\max_i\frac{\omega_i}{\omega_{i+1}}\cdot\max_i\frac{\omega_{i+1}}{\omega_i}}$, $D=\max_i\ \left\|P_{i+1}-P_i\right\|$ .

Deng and Li [9] obtained a new inequality of the rational Bézier curves as

$$\left\|R'(t)\right\| \le n\omega^{(2m+1)/3}\max_{0\le i\le n-1}\ \left\|P_{i+1}-P_i\right\|\ . \tag{2.7}$$

Jin [16] provided an estimation method for the rational conic Bézier curves. They proved that the bound can be defined as

$$\left\|R'(t)\right\| \le \begin{cases}\frac{\omega_1}{m}\cdot 2d\ \frac{\omega_0\omega_2}{\omega_1^2}\le 1,\ \omega_1\ge m\\ \sqrt{\frac{M}{m}}\cdot 2d\ \frac{\omega_0\omega_2}{\omega_1^2}>1,\ \omega_1\ge m\\ \frac{\sqrt{M}(\sqrt{\omega_0}+\sqrt{\omega_2})}{\omega_1+\sqrt{\omega_0\omega_2}}\cdot 2d,\ \omega_1<m\end{cases}\ , \tag{2.8}$$

where $d=\max_{0\le i\le 1}\ \left\|P_{i+1}P_i\right\|$ , $M=\max\ \omega_0,\omega_2$ , $m=\min\ \omega_0,\omega_2$ .

Very recently, Wang [17] gave the following bound of a NURBS curve of order $k$ (degree $k-1$) as

$$\left\|R'(t)\right\| \le (k-1)(\frac{\max_{1\le i\le n}\omega_i}{\min_{1\le i\le n}\omega_i})^2\ \frac{\max_{1\le i\le n}\left\|P_{i+1}-P_i\right\|}{\min_{1\le i\le n}(t_{i+k}-t_{i+1})},\ n\ge k\ , \tag{2.9}$$

There are various derivative bounds of the rational Bézier curves. We consider these bounds with drastically changing weights and control points. Then, we found that the estimated bounds of these methods are much larger than the exact bounds. The above methods are only applicable when the weights change smoothly. The key to the algorithm in this paper is how to suppress the influence of the weights and control points on the upper bounds of the derivative of the rational Bézier curves.

## 3 METHOD

### 3.1 Preliminaries

In this section, some novel conditions for estimating the bounds are presented. We present and prove the following Theorem.

**Theorem 1**

General polynomials and Bernstein polynomials can be transformed into each other.

**Proof.**

Let $p(x)=a_0+a_1x+\cdots+a_nx^n$ have real coefficients, Cargo and Shisha [18] note that $p(x)$ has a representation in Bernstein form, namely

$$p(x)=\sum_{j=0}^{n}b_j\binom{n}{j}x^j(1-x)^{n-j}\ , \tag{3.1}$$

The numbers $b_0, b_1 \cdots b_n$ are determined by $b_j = \sum_{i=0}^{j} a_0 \frac{\binom{j}{i}}{\binom{n}{i}},\ j = 0,1,\cdots n$.

Similarly, let $q(x) = \sum_{i=0}^{j} b_j \binom{n}{j} x^j (1-x)^{n-j}$ have real coefficients we can obtain $a_n$ by solving the equations. The equations are determined by

$$\begin{cases} b_0 = a_0 \cdot \frac{C_0^0}{C_n^0} \\ b_1 = a_0 \cdot \frac{C_1^0}{C_n^1} + a_1 \cdot \frac{C_1^1}{C_n^1} \\ \vdots \\ b_n = a_0 \cdot \frac{C_n^0}{C_n^n} + a_1 \cdot \frac{C_n^1}{C_n^n} + \cdots + a_n \cdot \frac{C_n^n}{C_n^n} \end{cases} \tag{3.2}$$

By (3.1)-(3.2), Theorem 1 holds.

### 3.2 Estimating the bounds on the magnitude of derivatives of rational Bézier curves

**Theorem 2**

For the rational Bézier curve defined by (1.1), we have $\left\| R'(t) \right\| \le \frac{\sqrt{\delta_x^2 + \delta_y^2 + \delta_z^2}}{\sigma^2}$, where $\delta_x = \max_{t \in [0,1]} \left\{ n \left[ \sum_{i=0}^{n-1} B_i^{n-1}(t)(\omega_{i+1} x_{i+1} - \omega_i x_i) \right] \cdot (\sum_{i=0}^{n} B_i^n(t)\omega_i) - (\sum_{i=0}^{n} B_i^n(t)\omega_i x_i) \cdot n \left[ \sum_{i=0}^{n-1} B_i^{n-1}(t)(\omega_{i+1} - \omega_i) \right] \right\}, t \in [0,1]$. $x_i$ is the x-component, $y_i$ is the y-component, $z_i$ is the z-component of the control point $P_i$, $\sigma = \min_{t \in [0,1]} (\sum_{i=0}^{n} B_i^n(t)\omega_i)$. $\delta_x$, $\delta_x$, $\delta_x$ and $\sigma$ respectively correspond to the parameter $t$ on the same interval.

**Proof.**

From (1.1), we have

$$R'(t) = \frac{(\sum_{i=0}^{n} B_i^n(t)\omega_i P_i)' \cdot (\sum_{i=0}^{n} B_i^n(t)\omega_i) - (\sum_{i=0}^{n} B_i^n(t)\omega_i P_i) \cdot (\sum_{i=0}^{n} B_i^n(t)\omega_i)'}{(\sum_{i=0}^{n} B_i^n(t)\omega_i)^2}, \tag{3.3}$$

According to the properties of Bernstein basis functions, we have

$$B_i^n(t) = \begin{cases} \binom{n}{t} t^i (1-t)^{n-i} & i = 0,1,\cdots n \\ 0 & \text{otherwise} \end{cases} \tag{3.4}$$

$$\frac{dB_i^n(t)}{dt} = n\left[B_{i-1}^{n-1}(t) - B_i^{n-1}(t)\right] \tag{3.5}$$

By (3.3)-(3.5) we have

$$\begin{aligned}
&(\sum_{i=0}^{n} B_i^n(t)\omega_i P_i)' = (B_0^n(t)\omega_0 P_0 + B_1^n(t)\omega_1 P_1 + \cdots + B_n^n(t)\omega_n P_n)' \\
&= n\left[B_{-1}^{n-1}(t) - B_0^{n-1}(t)\right]\omega_0 P_0 + n\left[B_0^{n-1}(t) - B_1^{n-1}(t)\right]\omega_1 P_1 + \cdots + n\left[B_{n-1}^{n-1}(t) - B_n^{n-1}(t)\right]\omega_n P_n \text{ ,} \\
&= n\left[\sum_{i=0}^{n-1} B_i^{n-1}(t)(\omega_{i+1}P_{i+1} - \omega_i P_i)\right]
\end{aligned}$$

Similarly, $(\sum_{i=0}^{n} B_i^n(t)\omega_i)' = n\left[\sum_{i=0}^{n-1} B_i^{n-1}(t)(\omega_{i+1} - \omega_i)\right]$. Then we have

$$R'(t) = \frac{n\left[\sum_{i=0}^{n-1} B_i^{n-1}(t)(\omega_{i+1}P_{i+1} - \omega_i P_i)\right]\cdot(\sum_{i=0}^{n} B_i^n(t)\omega_i) - (\sum_{i=0}^{n} B_i^n(t)\omega_i P_i)\cdot n\left[\sum_{i=0}^{n-1} B_i^{n-1}(t)(\omega_{i+1} - \omega_i)\right]}{(\sum_{i=0}^{n} B_i^n(t)\omega_i)^2} . \tag{3.6}$$

For brevity, we decompose $R'(t)$ into $A(t)$, $A_1(t)$, $A_2(t)$, $B(t)$, $B_1(t)$, $B_2(t)$ and $C(t)$ as follows:

$$R'(t) = \frac{n\cdot A(t) - n\cdot B(t)}{C(t)} , \tag{3.7}$$

where $A_1(t) = \sum_{i=0}^{n-1} B_i^{n-1}(t)(\omega_{i+1}P_{i+1} - \omega_i P_i)$ , $A_2(t) = \sum_{i=0}^{n} B_i^n(t)\omega_i$ , $A(t) = A_1(t)\cdot A_2(t)$ , $B_1(t) = \sum_{i=0}^{n} B_i^n(t)\omega_i P_i$ , $B_2(t) = \sum_{i=0}^{n-1} B_i^{n-1}(t)(\omega_{i+1} - \omega_i)$ , $B(t) = B_1(t)\cdot B_2(t)$ , $C(t) = (\sum_{i=0}^{n} B_i^n(t)\omega_i)^2$ . Take the 2-norm of the space vector, we have $\left\|A(t) - B(t)\right\| = \left\|A_1(t)\cdot A_2(t) - B_1(t)\cdot B_2(t)\right\| = \sqrt{(A^x(t) - B^x(t))^2 + (A^y(t) - B^y(t))^2 + (A^z(t) - B^z(t))^2}$ ,

where $A^x(t) = (\sum_{i=0}^{n-1} B_i^{n-1}(t)(\omega_{i+1}x_{i+1} - \omega_i x_i))\cdot(\sum_{i=0}^{n} B_i^n(t)\omega_i)$ is the x-component of $A(t)$ , $A^y(t)$ is the y-component of $A(t)$, and $A^z(t)$ is the z-component of $A(t)$. Similarly, $B^x(t)$ is the x-component of $B(t)$, $B^y(t)$ is the y-component of $B(t)$, $B^z(t)$ is the z-component of $B(t)$.

For the rational Bézier curve of degree $n$ , the Bernstein polynomial $(A^x(t) - B^x(t))^2$ can be transformed into a uniform Bernstein polynomial of degree $4n-2$ . The same transformation method was applied to $(A^y(t) - B^y(t))^2$ and $(A^z(t) - B^z(t))^2$ . Based on the extreme value characteristic of the Bernstein basis function, the interval of the parameter $t$ of $(A^x(t) - B^x(t))^2$ should be equally divided into $4n-2$ parts to estimate the extreme value. Similarly, the interval of $C(t)^2$ should be equally divided into $2n$ parts. We set $\delta_x = \max_{t\in[0,1]} A^x(t) - B^x(t)$ , $\delta_y = \max_{t\in[0,1]} A^y(t) - B^y(t)$ ,

$\delta_z = \max_{t\in[0,1]} A^z(t) - B^z(t)$ . To get the maximum value of $\|A(t) - B(t)\|$ , we first consider obtaining the extremum value of $(A^x(t) - B^x(t))^2$ . According to Theorem 1, we have the Bernstein form of polynomial $(A^x(t) - B^x(t))^2$, then we transform the Bernstein form into a unified general form of $(A^x(t) - B^x(t))^2$ . We define the general form of $(A^x(t) - B^x(t))^2$ as $Gen_{Ax-Bx}$ . To estimate the extremum value, we transform $Gen_{Ax-Bx}$ into a unified Bernstein form called $Bez_{Ax-Bx}$ . Similarly, we can obtain the unified Bernstein form of y-component and z-component which called $Bez_{Ay-By}$ and $Bez_{Az-Bz}$ . We use the same method to estimate the minimum value of $C(t)^2$ , whose Bernstein form is $Bez_C$ .

The degree of $(Bez_{Ax-Bx} + Bez_{Ay-By} + Bez_{Az-Bz})$ is $4n-2$ , and the degree of $Bez_C$ is $2n$ . The parameter interval is divided into $6n-2-m$ segments, where $m$ is the number of repeated node values. Bernstein basis functions of $(Bez_{Ax-Bx} + Bez_{Ay-By} + Bez_{Az-Bz})$ and $Bez_C$ are monotonic over each interval segment. We estimate the extreme values of $\|A(t) - B(t)\|$ and $C(t)^2$ on each interval, and calculate the upper bound of the derivative by (20) and (21). Take the maximum value of all the estimated upper bounds as the final result and define $\sigma = \min_{t\in[0,1]}(\sum_{i=0}^{n} B_i^n(t)\omega_i)$ . Then we have the sharp bounds

$$\|R'(t)\| \le \frac{\sqrt{\delta_x^2 + \delta_y^2 + \delta_z^2}}{\sigma^2}. \tag{3.8}$$

By (3.3)-(3.8), Theorem 2 holds.

## 4 RESULTS AND DISCUSSIONS

This section presents some numerical experiments based on the proposed algorithm and discusses the comparisons of multiple solutions.

### 4.1 A numerical Example

An example is presented to illustrate the superiority of the proposed results. Control points $P_i(i = 0,1,2,3)$ and weights $\omega_i(i = 0,1,2,3)$ for the example curve $R_0(t)$ : $P_0(19,61,0)$ , $P_1(-61,52,0)$ , $P_2(17,55,0)$ , $P_3(49,-20,0)$ , $\omega_0 = 1.7$ , $\omega_1 = 0.5$ , $\omega_2 = 1$ , $\omega_3 = 6.7$ .
The Bernstein basis function is shown in Figure 1.

It can be obtained that the interval has been divided into 14 segments to estimate the extreme value. The segments are $[0,0.1]$ , $[0.1,0.166]$ , $[0.166,0.2]$ , $[0.2,0.3]$ , $[0.3,0.333]$ , $[0.333,0.4]$ , $[0.4,0.5]$, $[0.5,0.6]$, $[0.6,0.666]$, $[0.666,0.7]$, $[0.8,0.833]$, $[0.833,0.9]$, $[0.9,1]$. The Bernstein basis function is monotonic in each interval, and the extreme value can be calculated by calculating the value of the start and end points in each interval. By simple calculation, we can derive that the estimated bound is 303.69. From Theorem 2, the derivative of the example rational Bézier curve has the following result: $\|R_0{}'(t)\| \le 303.69$ .

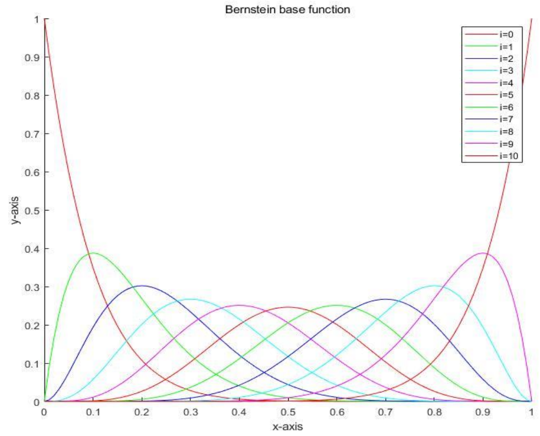


**Figure 1**: The Bernstein basis function of $(A(t)-B(t))^2$ .

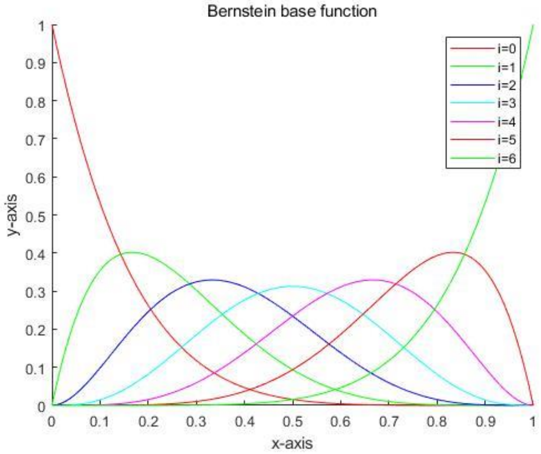


**Figure 2**: The Bernstein basis function of $(C(t))^2$ .

Then we compare our new bound with many other bounds. All of the cases of rational Bézier curves ( $R_i(t), i=1,2,\cdots 8$ )are provided in Appendix.

| *Curve* | *Exact* | *New bound* | *Floater* | *Selimovic* | *Huang* | *Deng* | *Li* | *Jin* | *Wang* |
|---|---|---|---|---|---|---|---|---|---|
| $R_0(t)$ | 181.41 | 303.69 | 43924 | 73573 | 2455 | 1203 | 578.79 | - | 43924 |
| $R_1(t)$ | 5.45 | 9.299 | 2000 | 2000 | 200 | 200 | 28.28 | - | 2000 |
| $R_2(t)$ | 2.797 | 4.02 | 2700 | 81000 | 125.17 | 42.9 | 30.06 | - | 2700 |
| $R_3(t)$ | 41.77 | 55.17 | 3.4E+7 | 1.6E+8 | 2668.6 | 6363.5 | 847.02 | - | 3.4E+7 |
| $R_4(t)$ | 309.3 | 573.4 | 4.52E+7 | 5.6E+9 | 14791 | 61476 | 2067.02 | - | 4.52E+7 |
| $R_5(t)$ | 328.0 | 523.4 | 3.2E+8 | 1.1E+15 | 72729 | 41826 | 9598 | - | 3.2E+8 |
| $R_6(t)$ | 90.63 | 159.5 | 63818 | 5.63E+10 | 5.63E+4 | 2.07E+5 | 2.55E+4 | - | 63818 |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $R_7(t)$ | 27.08 | 40.76 | 1.4E+6 | 3.4E+13 | 2917.5 | 42081 | 291.29 | - | 1.4E+6 |
| $R_8(t)$ | 201.2 | 455.1 | 1006.2 | 1006.2 | 201.2 | 188.25 | 201.24 | 4.55 | 1006.2 |

**Table 1**: The estimation bounds of different methods.

Huang and Zhu [20] proposed a fast point-by-point generating algorithm for rational para-metric curves. The core step of the algorithm needs to use the derivative bound of the curve to determine the iteration step size. Our new bound is closer to the exact one, minimizing the size of iterations and thus saving much calculation time. Figure 3 shows the drawing result of curve $R_0(t)$, and the calculation time results are displayed in Figure 4. In Zheng's manuscript [21], the estimation bound influences the step size of rational Bézier curves. We ran several numerical comparisons on test step size cases of randomly generated rational Bézier curves. The results are displayed in Figure 5. It is worth noting that the calculation time and step size are set to 1 with the exact boundary value, and other estimation results are presented as ratios. The curves are numbered 1-9 to present curves $R_0(t)$ - $R_8(t)$.

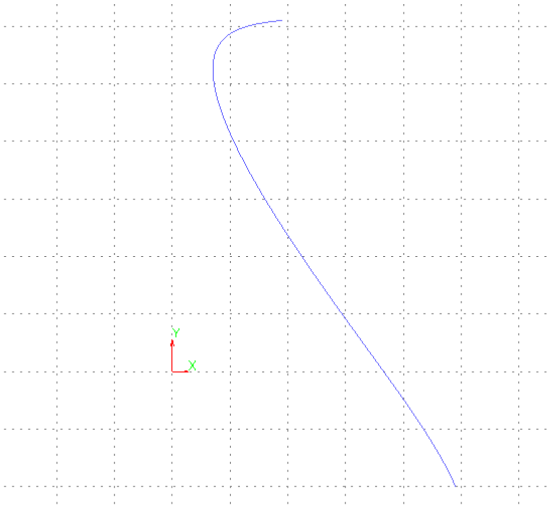


**Figure 3**: The drawing result of $R_0(t)$.

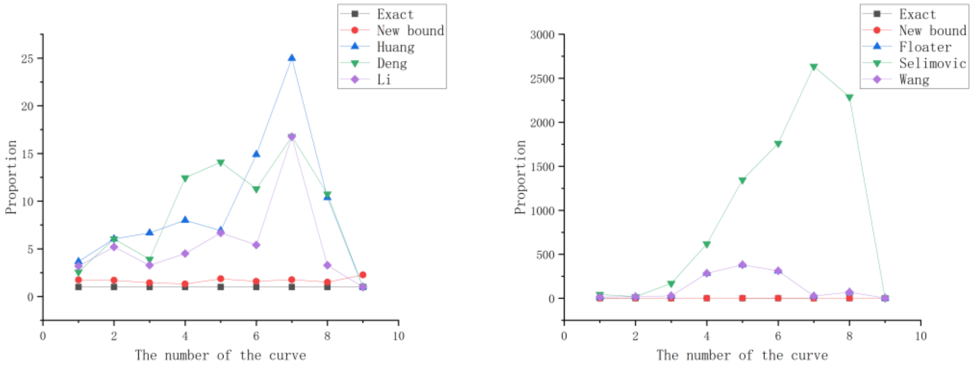


**Figure 4**: The calculation time results.

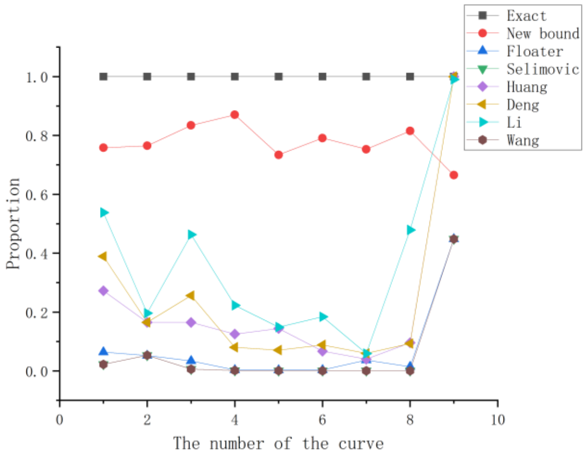


**Figure 5**: The step size results.

## 4.2 Comparisons and Discussions

There are many derivative bounds of rational Bézier curves. We compare Theorem 3.1 with the estimations proposed by Floater [7], Selimovic [9], Huang [10], Deng [12], Li [13], Jin [16] and WGJ [17]. We compare results for the bounds in higher degree cases on the rational Bézier curves.

We compare our new bounds with many other bounds by numerical examples, which are summarized in Table 1, and the remaining results are similar, hence are not reported here. Since the method of Jin [16] only applies to rational conic Bézier curves, it is invalid in some numerical examples. These results show that our new bound is generally better than those proposed in the current literature. The curve generation time is significantly reduced, and the generated step size within the tolerance range is as large as possible to benefit the mesh generation algorithm.

The proposed method has an excellent convergence effect on the typical conic section curve and the standard Bézier curve in engineering, significantly when the local weight factor changes drastically. The estimation methods proposed in the current literature have huge estimation errors for this situation. In the current estimation method, the control points and the weights are separated to estimate the derivative value of the rational Bézier curve. Weights usually appear in the denominator, and the control points appear in the numerator. However, this ignores the influence of the local change rate of the control points and weight factors on the estimation bounds. When the curve is smooth, and the local control points and weights change drastically, it will cause significant errors in the estimation. The method we propose is based on the characteristics of Bernstein basis functions, and the product of the control points and the weight factors are used as a whole part to estimate the derivative value in sections, thereby avoiding this problem.

Li and Deng’s estimation [13] is the same as Huang’s [10]. However, this kind of curve is scarce in the practical application of engineering, and our estimated result is less different from the precise value. In general, our method shows better results than Li and Deng's.

## 5 CONCLUSIONS

This paper presents a novel method for estimating the bounds of the derivatives of rational Bézier curves using Bernstein basis functions' characteristics and fundamental inequalities. Theoretical analysis and numerical examples validate that our new bounds are tighter than the existing ones for smooth rational Bézier curves with drastic weights and control points. In addition, the new bound is less affected by changes in weights and control points, proving our method's superiority. Furthermore, the proposed method solves the problem that the existing bounds in the literature will cause significant errors when the weights change considerably. It has an excellent convergence effect for smooth but local control vertices and severe changes in weight factors, which is beneficial for studies on computer graphics and complex geometric modeling. In the following works, we can follow the derivation in the manuscript and extend the method to the upper bound estimation of the second-order derivative of rational Bézier curves.

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## APPENDIX

This part provides the cases of rational Bézier curves. The control point is represented here as $(x\omega, y\omega, z\omega, \omega)$ .It should be noted that the x, y and z coordinate values are the products of the original coordinate values and the weights.

$R_1(t)$ : control points $P = (0,0,0,1),(0,1,0,0.1),(2,2,0,1)$

$R_2(t)$ : control points $P = (0,0,0,1),(0,1,0,0.1),(0.1,0.1,0,0.1),(3,0,0,3)$

$R_3(t)$ : control points $P = \begin{Bmatrix} (6.1,30.5,0,6.1),(2.73,2.73,0,0.39),(-0.24,-0.3,0,0.03),(-1.1,-11,0,1.1), \\ (-110.4,-55.2,0,18.4) \end{Bmatrix}$

$R_4(t)$ : control points $P = \begin{Bmatrix} (7791,-882,0,147),(-46.2,435.6,0,6.6),(-44.8,-32.2,0,0.7),(-131.4,77.4,0,1.8), \\ (67.9,-47.6,0,0.7),(-264,228,0,4) \end{Bmatrix}$

$R_5(t)$ : control points $P = \begin{Bmatrix} (61.2,-39.1,0,1.7),(38.4,43.2,0,0.8),(2.8,-2.6,0,0.2),(6.4,1.3,0,0.1), \\ (-27.2,21.6,0,0.4),(25.8,-0.6,0,0.6),(2142,5796,0,63) \end{Bmatrix}$

$R_6(t)$ : control points $P = \begin{Bmatrix} (38.7,38.7,0,4.3),(-10.4,0,0,2.6),(1,0,0,0.2),(-9.1,0,0,1.3), \\ (4,-4,0,0.4),(1.2-1.8,0,0.6),(-10.8,24.3,0,2.7),(3.1,-15.5,0,3.1) \end{Bmatrix}$

$R_7(t)$ : control points $P = \begin{Bmatrix} (117,78,0,39),(-11.5,18.4,0,2.3),(0.96,3.84,0,0.96),(12.6,18,0,1.8), \\ (0.4,-3.2,0,0.4),(-4.2,3,0,0.6),(0,13,0,1.3),(2.8,0.7,0,0.7),(110.4,-18.4,0,18.4) \end{Bmatrix}$

$R_8(t)$ : control points $P = (-0.9,-3,0,0.3),(9,12,0,1.5),(1.2,2.4,0,0.6)$

## REFERENCES

[1] Lin, H.-W.: On the derivative formula of a rational Bézier curve at a corner, Applied Mathematics and Computation, 210(1), 2009, 197–201. http://doi.org/10.1016/j.amc.2008.12.078

[2] Wu, Z.-K.; Lin, F.; Soon, S.-H.; and Yun, C.-K.: Evaluation of difference bounds for computing rational Bézier curves and surfaces, Computers and Graphics, 28(4), 2004, 551–558. http://doi.org/10.1016/j.cag.2004.04.010

[3] Abi, E.-S.; Shirman, L.A.: Tessellation of curved surfaces under highly varying transformations, Eurographics, 9(2), 1991, 385–397. http://dx.doi.org/10.2312/egtp.19911028
[4] Sheng, X.; Hirsch, B.E.: Triangulation of trimmed surfaces in parametric space, Computer Aided Design, 24(8), 1992, 437–444. https://doi.org/10.1016/0010-4485(92)90011-X
[5] Piegl, L.; Richard, A.M.: Tessellating trimmed NURBS surfaces, Computer Aided Design, 27(1), 1995, 16–26. https://doi.org/10.1016/0010-4485(95)90749-6
[6] Filip, D.; Magedson,R.; Markot, R.: Surface algorithms using bounds on derivatives, Computer Aided Geometric Design, 3(4), 1986, 295–311. https://doi.org/10.1016/0167-8396(86)90005-1
[7] Floater, M.-S.: Derivatives of rational Bézier curves, Computer Aided Geometric Design, 9(3),1992, 161–174. http://doi.org/10.1016/0167-8396(92)90014-G
[8] Hermann, T.: On the derivatives of second and third degree rational Bézier curves, Computer Aided Geometric Design, 16(3), 1999, 157–163. http://doi.org/10.1016/S0167-8396(98)00037-5
[9] Selimovic, I.; New bounds on the magnitude of the derivative of rational Bézier curves and surfaces, Computer Aided Geometric Design, 22(4), 2005, 321–326. http://doi.org/10.1016/j.cagd.2005.01.003
[10] Huang, Y.; Su, H.: The bound on derivatives of rational Bézier curves, Computer Aided Geometric Design, 23(9), 2006, 698–702. http://doi.org/10.1016/j.cagd.2006.08.001
[11] Zhang, R.-J; Ma, W.: Some improvements on the derivative bounds of rational Bézier curves and surfaces, Computer Aided Geometric Design, 23(7), 2006, 563–572. http://doi.org/10.1016/j.cagd.2006.01.006
[12] Deng, C.-Y.: Improved bounds on the magnitude of the derivative of rational Bézier curves, Applied Mathematics and Computation, 218(1), 2011, 204–206. http://doi.org/10.1016/j.amc.2011.05.068
[13] Deng, C.-Y., Li, Y.: A new bound on the magnitude of the derivative of rational Bézier curve, Computer Aided Geometric Design, 30(2), 2013, 175–180. https://doi.org/10.1016/j.cagd.2012.11.002
[14] Bez, H.-E, Bez, N.: On derivative bounds for the rational quadratic Bézier paths, Computer Aided Geometric Design, 30(2), 2013, 254–261. http://doi.org/10.1016/j.cagd.2012.12.003
[15] Bez, H.-E, Bez, N.: New minimal bounds for the derivatives of rational Bézier paths and rational rectangular Bézier surfaces, Applied Mathematics and Computation, 225(1), 2013, 475–479. http://doi.org/10.1016/j.amc.2013.09.039
[16] Jin, W., Deng, C.-Y., Li, Y.-J, Liu, J.-Z: Derivative bound estimations on rational conic Bézier curves, Applied Mathematics and Computation, 248(1), 2014, 113-117. http://doi.org/10.1016/j.amc.2014.09.121
[17] Wang, G.-J., Xu, H., Hu, Q.: Bounds on partial derivatives of NURBS surfaces, Applied Mathematics-A Journal of Chinese Universities, 32(1), 2017, 281-293. https://doi.org/10.1007/s11766-017-3429-0
[18] Cargo, G.-T., Shisha, O.: The Bernstein form of a polynomial, Journal of Research of the National Bureau of Standards, Section B, 70(B), 1966, 79-81. http://doi.org/10.6028/jres.070B.005
[19] Xie, B.-H, Wang, G.-J.: Approximating the Derivative Bounds of Parametric Curves and Applying to Curve Rasterization, Journal of Software, 14(12), 2003, 2106-2112
[20] Huang, Y.-D, Zhu, G.-Q.: A Fast Point-by-Point Generating Algorithm for Rational Parametric Curves, Chinese Journal of Computers, 7(8), 2001, 809-814.
[21] Sederberg, T.-W. and Zheng, J.-M: Estimating Tessellation Parameter Intervals for Rational Curves and Surfaces, ACM Transactions on Graphics, 19(1), 2000, 56-77. https://doi.org/10.1145/343002.343034