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Computer-Aided Design AND APPLICATIONS

Application of the Shape Uniqueness Theorem to the H-Bézier Curve

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Abstract. The shape uniqueness theorem for free-form curves shows the conditions on which the shapes of two parametric curves defined by three control points are identical altough their parametrization may be different [6]. According to this theorem, even though their blending functions look different, the curves become identical by reparametrization under some conditions on their blending functions.

A lot of researches have been done on the blending functions of free-form curves so far and many types of free-from curves are available for curve designers. These designers must be confused on which type of curve should be used for their design. We hope that the shape uniqueness theorem for free-form curves will help the designers classify and categorize types of curves and select the most suitable one for their design purposes because it identifies the curves which superficially look different but represent the same shape.

In this paper we will apply the shape uniqness therem to the H-Bézier curve, whose blending functions are defined by recursively using integral forms. We think that it is worth while to apply the shape uniqness theorem to a uniquely defined free-form curve.

Keywords: shape uniqueness theorem, H-Bézier curve, reparametrization
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1 INTRODUCTION

The shape uniqueness theorem for free-form curves shows the conditions on which the shapes of two parametric curves defined by three control points are identical altough their parametrization may be different. According to this theorem, even though their blending functions look different, the curves become identical by reparametrization under some conditions on their blending functions.

A lot of researches have been done on the blending functions of free-form curves so far and many types of free-from curves are available for curve designers. These designers must be confused on which type of curve should be used for their design. We hope that the shape uniqueness theorem for free-form curves will help the designers classify and categorize types of curves and select the most suitable one for their design purposes because it identifies the curves which superficially look different but represent the same shape.

In this paper we will apply the shape uniquness therem to the H-Bézier curve [3], whose blending functions are defined by recursively using integral forms. We think that it is worth while to apply the shape uniquness theorem to a uniquely defined free-form curve.

2 H-BÉZIER CURVE [3, 2]

The H-Bézier curve of degree n with parameter α is

$$\boldsymbol{q}(t) = \sum_{i=0}^{n} Z_i^n(t)\boldsymbol{b}_i \tag{1}$$

for $t \in [0,1]$, where Z_i^n is *the H-basis function* of degree n defined by

$$Z_0^1(t) = \frac{\sinh \alpha(1-t)}{\sinh \alpha} \tag{2}$$

$$Z_1^1(t) = \frac{\sinh \alpha t}{\sinh \alpha} \tag{3}$$

and recursively

$$W_i^n(t) = \int_0^t Z_i^n(s)ds \quad (0 \le i \le n) \tag{4}$$

$$Z_0^{n+1}(t) = 1 - \frac{W_0^n(t)}{W_0^n(1)} \tag{5}$$

$$Z_i^{n+1}(t) = \frac{W_{i-1}^n(t)}{W_{i-1}^n(1)} - \frac{W_i^n(t)}{W_i^n(1)} \quad (1 \le i \le n) \tag{6}$$

$$Z_{n+1}^{n+1}(t) = \frac{W_n^n(t)}{W_n^n(1)} \tag{7}$$

for $n \ge 1$. The quadratic H-basis functions are

$$Z_0^2(t) = \frac{1-\cosh \alpha(1-t)}{1-\cosh \alpha} \tag{8}$$

$$Z_1^2(t) = \frac{\cosh \alpha(1-t) - \cosh \alpha - 1 + \cosh \alpha t}{1-\cosh \alpha} \tag{9}$$

$$Z_2^2(t) = \frac{1-\cosh \alpha t}{1-\cosh \alpha} \tag{10}$$

and the cubic H-basis functions are

$$Z_0^3(t) = \frac{\alpha(1-t) - \sinh\alpha(1-t)}{\alpha - \sinh\alpha} \tag{11}$$

$$Z_1^3(t) = \frac{\alpha t + \sinh\alpha(1-t) - \sinh\alpha}{\alpha - \sinh\alpha} - \frac{\sinh\alpha(1-t) + \alpha t\cosh\alpha + \alpha t - \sinh\alpha t - \sinh\alpha}{\alpha\cosh\alpha + \alpha - 2\sinh\alpha} \tag{12}$$

$$Z_1^3(t) = \frac{\sinh\alpha(1-t) + \alpha t\cosh\alpha + \alpha t - \sinh\alpha t - \sinh\alpha}{\alpha\cosh\alpha + \alpha - 2\sinh\alpha} - \frac{\alpha(1-t) - \sinh\alpha t}{\alpha - \sinh\alpha} \tag{13}$$

$$Z_0^3(t) = \frac{\alpha(1-t) - \sinh\alpha t}{\alpha - \sinh\alpha} \tag{14}$$

Note that

$$\frac{Z_1^2(t)^2}{Z_0^2(t)Z_2^2(t)} = 2(1+\cosh\alpha). \tag{15}$$

The quadratic rational Bézier basis functions $R_i^2(t)$, $i = 0, 1, 2$ are given by

$$R_0^2(t) = \frac{(1-t)^2}{(1-t)^2 + 2(1-t)tw + t^2} \tag{16}$$

$$R_1^2(t) = \frac{2(1-t)tw}{(1-t)^2 + 2(1-t)tw + t^2} \tag{17}$$

$$R_2^2(t) = \frac{t^2}{(1-t)^2 + 2(1-t)tw + t^2} \tag{18}$$

where w is the weight of the second control point. Then

$$\frac{R_1^2(t)^2}{R_0^2(t)R_2^2(t)} = 4w^2 \tag{19}$$

From the shape uniqueness theorem for the free-form curve defined by three control points [6], the shapes of quadratic H-Bézier and rational Bézier curves are identical for the same given control points if $2(1+\cosh\alpha) = 4w_h^2$, i.e. the equivalent weight $w_h = \sqrt{\frac{1+\cosh\alpha}{2}}$. Since $\cosh\alpha > 1$ for $\alpha > 0$, $\sqrt{\frac{1+\cosh\alpha}{2}} > 1$.

2.1 Shape Equivalence of the Cubic H-Bézier Curve

In this section, we will discuss on the shapes of cubic H-Bézier curve and the up-degreed curve of the quadratic rational Bézier curve based on the recursive algorithm explained in the previous section. The up-degree procedure is different from the generation by multiplying $(1-t)+t$ to the lower basis functions. In the quadratic rational Bézier basis functions are up-degreed as

$$R_0^3(t)=\frac{\sqrt{-w-1}\sqrt{w-1}(\log(2t(t(-w)+t+w-1)+1)-2t+2)}{2\sqrt{-w-1}\sqrt{w-1}+4w\tan^{-1}\left(\frac{\sqrt{w-1}}{\sqrt{-w-1}}\right)}+\frac{2w\left(\tan^{-1}\left(\frac{(1-2t)\sqrt{w-1}}{\sqrt{-w-1}}\right)+\tan^{-1}\left(\frac{\sqrt{w-1}}{\sqrt{-w-1}}\right)\right)}{2\sqrt{-w-1}\sqrt{w-1}+4w\tan^{-1}\left(\frac{\sqrt{w-1}}{\sqrt{-w-1}}\right)} \quad (20)$$

$$R_1^3(t)=\frac{(w-1)\left((w+1)\log(2t(t(-w)+t+w-1)+1)+2\sqrt{-w-1}\sqrt{w-1}\tan^{-1}\left(\frac{(2t-1)\sqrt{w-1}}{\sqrt{-w-1}}\right)\right)}{2\left(\sqrt{-w-1}\sqrt{w-1}+2\tan^{-1}\left(\frac{\sqrt{w-1}}{\sqrt{-w-1}}\right)\right)\left(\sqrt{-w-1}\sqrt{w-1}+2w\tan^{-1}\left(\frac{\sqrt{w-1}}{\sqrt{-w-1}}\right)\right)}-\frac{2\sqrt{-w-1}\sqrt{w-1}\tan^{-1}\left(\frac{\sqrt{w-1}}{\sqrt{-w-1}}\right)((2t-1)(w-1)+\log(2t(t(-w)+t+w-1)+1))}{2\left(\sqrt{-w-1}\sqrt{w-1}+2\tan^{-1}\left(\frac{\sqrt{w-1}}{\sqrt{-w-1}}\right)\right)\left(\sqrt{-w-1}\sqrt{w-1}+2w\tan^{-1}\left(\frac{\sqrt{w-1}}{\sqrt{-w-1}}\right)\right)} \quad (21)$$

$$R_2^3(t)=\frac{(w-1)\left((w+1)\log(2t(t(-w)+t+w-1)+1)-2\sqrt{-w-1}\sqrt{w-1}\tan^{-1}\left(\frac{(2t-1)\sqrt{w-1}}{\sqrt{-w-1}}\right)\right)}{2\left(\sqrt{-w-1}\sqrt{w-1}+2\tan^{-1}\left(\frac{\sqrt{w-1}}{\sqrt{-w-1}}\right)\right)\left(\sqrt{-w-1}\sqrt{w-1}+2w\tan^{-1}\left(\frac{\sqrt{w-1}}{\sqrt{-w-1}}\right)\right)}+\frac{2\sqrt{-w-1}\sqrt{w-1}\tan^{-1}\left(\frac{\sqrt{w-1}}{\sqrt{-w-1}}\right)((2t-1)(w-1)-\log(2t(t(-w)+t+w-1)+1))}{2\left(\sqrt{-w-1}\sqrt{w-1}+2\tan^{-1}\left(\frac{\sqrt{w-1}}{\sqrt{-w-1}}\right)\right)\left(\sqrt{-w-1}\sqrt{w-1}+2w\tan^{-1}\left(\frac{\sqrt{w-1}}{\sqrt{-w-1}}\right)\right)}$$

$$R_3^3(t)=\frac{\sqrt{-w-1}\sqrt{w-1}(\log(2t(t(-w)+t+w-1)+1)+2t)}{2\sqrt{-w-1}\sqrt{w-1}+4w\tan^{-1}\left(\frac{\sqrt{w-1}}{\sqrt{-w-1}}\right)}+\frac{2w\left(\tan^{-1}\left(\frac{(2t-1)\sqrt{w-1}}{\sqrt{-w-1}}\right)+\tan^{-1}\left(\frac{\sqrt{w-1}}{\sqrt{-w-1}}\right)\right)}{2\sqrt{-w-1}\sqrt{w-1}+4w\tan^{-1}\left(\frac{\sqrt{w-1}}{\sqrt{-w-1}}\right)} \quad (22)$$

These basis functions are different from those of the cubic Bézier curve and, for example $R_1^3(t)^2/(R_0^3(t)R_2^3(t))$ is dependent on parameter t. Hence they are quite different from those of the cubic rational Bézier curve. Please refer to [4] on the shape uniqueness theorem for the free-form crve defined by four or more control points. Figure 1 shows these basis functions with $w=1/2$.

2.2 Reparametrization of Integration

Identical shape of two parametric curves is defined as follows [1]:

Definition 1. *For two parametric curves $\boldsymbol{r}:I\rightarrow R^3$ and $\tilde{\boldsymbol{r}}:\tilde{I}\rightarrow R^3$, there exists a C^∞ function $\phi:I\rightarrow\tilde{I}$,* 1) *$\phi$ is a one to one and onto mapping from I to $\tilde{I}$.* 2) *ϕ is strictly increasing.* 3) *For all $t\in I$, $\tilde{\boldsymbol{r}}(\phi(t))=\boldsymbol{r}(t)$. We say that $\boldsymbol{r}$ and $\tilde{\boldsymbol{r}}$ define the same curve or their shapes are identical.*

Then $\tilde{\boldsymbol{r}}((\phi(t))$ is called reparametrization of $\boldsymbol{r}(t)$. For example, there is a function $\phi(t)$ such that

$$R_i^2(\phi(t))=Z_i^2(t) \quad i=0,1,2 \quad (23)$$

We assume that

$$\int_0^{\phi(t)}R_0^2(\phi(t))d\phi(t)=T_0^3(\phi(t)) \quad (24)$$

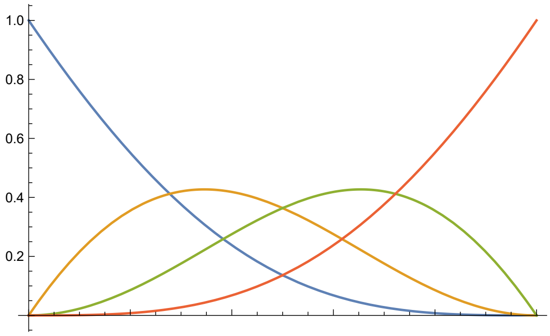

Figure 1: Basis functions of the up-degreed rational quadratic Bézier curve with $w = 1/2$.

Since $\phi(1) = 1$,

$$R_0^3(\phi(t)) = 1 - \frac{T_0^3(\phi(t))}{T_0^3(1)} \tag{25}$$

$$R_1^3(\phi(t)) = \frac{T_0^3(\phi(t))}{T_0^3(1)} - \frac{T_1^3(\phi(t))}{T_1^3(1)} \tag{26}$$

$$R_2^3(\phi(t)) = \frac{T_1^3(\phi(t))}{T_1^3(1)} - \frac{T_2^3(\phi(t))}{T_2^3(1)} \tag{27}$$

$$R_3^3(\phi(t)) = \frac{T_2^3(\phi(t))}{T_2^3(1)} \tag{28}$$

Hence, R_i^3 is naturally reparametrized by $\phi(t)$.

Before discussing the relationships among Z_i^3 and R_i^3, $i = 0, \cdots, 3$, we show the properties of Z_i^3. From their definition,

$$\sum_{i=0}^{3} Z_i^3(t) = 1 \tag{29}$$

$$\sum_{i=0}^{3} \frac{dZ_i^3(t)}{dt} = 0 \tag{30}$$

$$\sum_{i=0}^{3} \int_0^t Z_i^3(t)dt = t \tag{31}$$

Especially when $t = 1$, we obtain

$$\sum_{i=0}^{3} \int_0^1 Z_i^3(t)dt = 1 \tag{32}$$

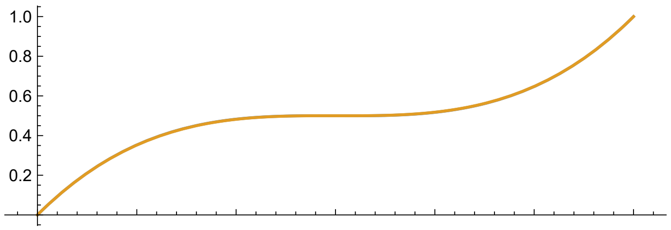

Figure 2: The identical two curves with $\alpha = 1/2$ and $w = \sqrt{\frac{1+\cosh\frac{1}{2}}{2}}$.

$Z_i^2(t)$ are given by

$$Z_0^2(t) = -\frac{dZ_0^3(t)}{dt}\int_0^1 Z_0^2(t)dt \tag{33}$$

$$Z_1^2(t) = \frac{dZ_0^3(t)}{dt}\int_0^1 Z_0^2(t)dt - \frac{dZ_1^3(t)}{dt}\int_0^1 Z_1^2(t)dt \tag{34}$$

$$Z_2^2(t) = \frac{dZ_1^3(t)}{dt}\int_0^1 Z_1^2(t)dt - \frac{dZ_2^3(t)}{dt}\int_2^1 Z_2^2(t)dt = \frac{dZ_3^3(t)}{dt}\int_2^1 Z_2^2(t)dt \tag{35}$$

Hence

$$\frac{Z_1^2(t)^2}{Z_0^2(t)Z_2^2(t)} = -\frac{(\frac{dZ_0^3(t)}{dt}\int_0^1 Z_0^2(t)dt - \frac{dZ_1^3(t)}{dt}\int_0^1 Z_1^2(t)dt)^2}{\frac{dZ_0^3(t)}{dt}\int_0^1 Z_0^2(t)dt\frac{dZ_3^3(t)}{dt}\int_2^1 Z_2^2(t)dt} \tag{36}$$

is not dependent on parameter t and a constant.

Since R_i^3 is naturally reparametrized by $\phi(t)$, given the same four control points, the shapes of the curves whose basis functions are $Z_i^3(t)$ and $R_i^3(t)$, respectively are identical. Figure 2 shows the identical two curves defined with the same control points $(0,0)$, $(1,1)$, $(2,0)$, and $(3,1)$.

3 Generalized Trigonometric Curve [5]

We will apply the recursive procedure explained in first section to obtain the quadratic generalized trigonometric basis functions. We will start from

$$\boldsymbol{q}(t) = \sum_{i=0}^{n} S_i^n(t)\boldsymbol{b}_i \tag{37}$$

for $t \in [0,1]$, where S_i^n are the basis functions of degree n defined by

$$S_0^1(t) = \cos\frac{\pi t}{2} \tag{38}$$

$$S_1^1(t) = \sin\frac{\pi 2}{2} \tag{39}$$

and recursively

$$Y_i^n(t) = \int_0^t S_i^n(s)ds \quad (0 \le i \le n) \tag{40}$$

$$S_0^{n+1}(t) = 1 - \frac{Y_0^n(t)}{Y_0^n(1)} \tag{41}$$

$$S_i^{n+1}(t) = \frac{Y_{i-1}^n(t)}{Y_{i-1}^n(1)} - \frac{Y_i^n(t)}{Y_i^n(1)} \quad (1 \le i \le n) \tag{42}$$

$$S_{n+1}^{n+1}(t) = \frac{Y_n^n(t)}{{}_n^n(1)} \tag{43}$$

for $n \ge 1$. The quadratic basis functions are

$$S_0^2(t) = 1 - \sin\frac{\pi t}{2} \tag{44}$$

$$S_1^2(t) = \sin\frac{\pi t}{2} + \cos\frac{\pi t}{2} - 1 \tag{45}$$

$$S_2^2(t) = 1 - \cos\frac{\pi t}{2} \tag{46}$$

The above basis functions are the same as the generalized trigonometric basis functions. Note that

$$\frac{S_1^2(t)^2}{S_0^2(t)S_2^2(t)} = 2 \tag{47}$$

Hence, from the shape uniqueness theorem for three control points, if the weight w of the second control point of a quadratic rational Bézier curve is equal to $1/\sqrt{2}$, the shapes of the quadratic generalized trigonometric curve and the quadratic rational Bézier one are identical.

The cubic basis functions are

$$S_0^3(t) = \frac{\pi(1-t) - 2\cos\left(\frac{\pi t}{2}\right)}{\pi - 2} \tag{48}$$

$$S_1^3(t) = \frac{-2\pi t + 2(\pi-2)\sin\left(\frac{\pi t}{2}\right) - 4\cos\left(\frac{\pi t}{2}\right) + 4}{(\pi-4)(\pi-2)} \tag{49}$$

$$S_2^3(t) = \frac{2\left(\pi(t-1) - 2\sin\left(\frac{\pi t}{2}\right) + (\pi-2)\cos\left(\frac{\pi t}{2}\right) + 2\right)}{(\pi-4)(\pi-2)} \tag{50}$$

$$S_3^3(t) = \frac{\pi t - 2\sin\left(\frac{\pi t}{2}\right)}{\pi - 2} \tag{51}$$

Figure 3 shows these cubic basis functions. They are different from the cubic generalized trigonometric basis functions and $S_1^3(t)^2/(S_0^3(t)S_2^3(t))$ is dependent on parameter t.

Since R_i^3 is naturally reparametrized by $\phi(t)$, given the same four control points, the shapes of the curves whose basis functions are $S_i^3(t)$ and $R_i^3(t)$, respectively are identical. Figure 4 shows the identical two curves defined with the same control points $(0,0)$, $(1,1)$, $(2,0)$, and $(3,1)$.

4 Extension of Generalized Trigonometric Curve [5]

We will apply the recursive procedure explained in first section to obtain an extension of the quadratic generalized trigonometric basis functions. Similar to Eqs.(2) and (3) we replace $\sinh$ with $\sin$, we will define S_0^1

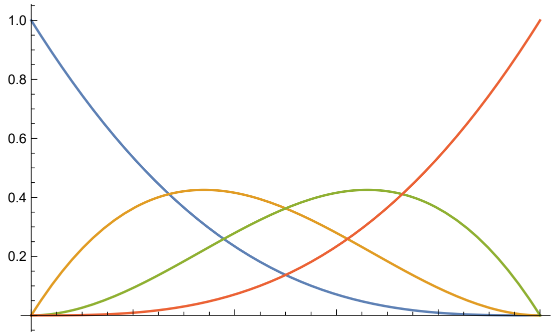

Figure 3: Basis functions of the up-degreed quadratic generalized trigonometric basis functions.

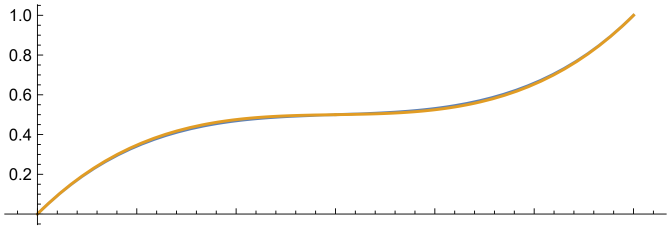

Figure 4: The identical two curves with $w = 1/\sqrt{2}$.

and S_1^1 as

$$S_0^1(t) = \frac{\sin \frac{\pi\beta(1-t)}{2}}{\sin \frac{\pi\beta}{2}} \tag{52}$$

$$S_1^1(t) = \frac{\sin \frac{\pi\beta\, t}{2}}{\sin \frac{\pi\beta}{2}} \tag{53}$$

and recursively

$$Y_i^n(t) = \int_0^t S_i^n(s)ds \quad (0 \le i \le n) \tag{54}$$

$$S_0^{n+1}(t) = 1 - \frac{Y_0^n(t)}{Y_0^n(1)} \tag{55}$$

$$S_i^{n+1}(t) = \frac{Y_{i-1}^n(t)}{Y_{i-1}^n(1)} - \frac{Y_i^n(t)}{Y_i^n(1)} \quad (1 \le i \le n) \tag{56}$$

$$S_{n+1}^{n+1}(t) = \frac{Y_n^n(t)}{{}_n^n(1)} \tag{57}$$

for $n \geq 1$. Note that to guarantee the monotonicity of $S_0^1(t)$ and $S_1^1(t)$, we restrict β as $0 < \beta \leq 1$. The quadratic basis functions are

$$S_0^2(t) = \frac{1 - \cos\frac{\pi\beta(1-t)}{2}}{1 - \cos\frac{\pi\beta}{2}} \tag{58}$$

$$S_1^2(t) = \frac{\cos\frac{\pi\beta(1-t)}{2} + \cos\frac{\pi\beta\, t}{2} - \cos\frac{\pi\beta}{2} - 1}{1 - \cos\frac{\pi\beta}{2}} \tag{59}$$

$$S_2^2(t) = \frac{1 - \cos\frac{\pi\beta\, t}{2}}{1 - \cos\frac{\pi\beta}{2}} \tag{60}$$

To guarantee the positivity of $S_0^2(t)$, $S_1^2(t)$ and $S_1^1(t)$, we restrict β as $0 < \beta \leq 2$. T The above basis functions are the same as the generalized trigonometric basis functions. Note that

$$\frac{S_1^2(t)^2}{S_0^2(t)S_2^2(t)} = 4\cos^2\frac{\pi\beta}{4} \tag{61}$$

From the shape uniqueness theorem for the free-form curve defined by three control points [6], the shapes of quadratic extended generalized trigonometric curves and rational Bézier curves are identical for the same given control points if $4\cos^2\frac{\pi b}{4} = 4w_t^2$, i.e. the equivalent weight $w_t = \cos\frac{\pi\beta}{4}$. Since $0 < \beta \leq 2$, $0 \leq \cos\frac{\pi b}{4} < 1$.

From the same theorem, the shapes of quadratic H-Bézier and extended trigonometric curves might be for the same given control points if their equivalent weights w_h and w_t are the same. However their equivalent weights won't be the same since $w_h = \sqrt{\frac{1+\cosh\alpha}{2}} > 1$ and $w_t = \cos\frac{\pi\beta}{4} < 1$.

The cubic basis functions are

$$S_0^3(t) = \frac{\pi\beta(1-t) - 2\sin\left(\frac{\pi\beta(1-t)}{2}\right)}{\beta\pi - 2\sin\frac{\pi\beta}{2}} \tag{62}$$

$$S_1^3(t) = \frac{\pi\beta t - 2\sin\left(\frac{1}{2}\pi\beta(t-1)\right) - 2\sin\left(\frac{\pi\beta}{2}\right)}{\pi\beta - 2\sin\left(\frac{\pi\beta}{2}\right)} + \frac{\csc\left(\frac{\pi\beta}{2}\right)\left(-\pi\beta t + 2\sin\left(\frac{\pi\beta t}{2}\right) + 2\sin\left(\frac{1}{2}\pi\beta(t-1)\right) + \pi(-\beta)t\cos\left(\frac{\pi\beta}{2}\right) + 2\sin\left(\frac{\pi\beta}{2}\right)\right)}{\pi\beta\cot\left(\frac{\pi\beta}{4}\right) - 4} \tag{63}$$

$$S_2^3(t) = \frac{-\pi\beta t + 2\sin\left(\frac{\pi\beta t}{2}\right) + \pi b - 2\sin\left(\frac{\pi\beta}{2}\right)}{\pi b - 2\sin\left(\frac{\pi\beta}{2}\right)} + \frac{\csc\left(\frac{\pi\beta}{4}\right)\sec\left(\frac{\pi\beta}{4}\right)\left(\pi\beta(t-1) - 2\sin\left(\frac{\pi\beta t}{2}\right) - 2\sin\left(\frac{1}{2}\pi\beta(t-1)\right) + \pi\beta(t-1)\cos\left(\frac{\pi\beta}{2}\right) + 2\sin\left(\frac{\pi\beta}{2}\right)\right)}{2\pi\beta\cot\left(\frac{\pi\beta}{4}\right) - 8} \tag{64}$$

$$S_3^3(t) = \frac{\pi\beta\, t - 2\sin\left(\frac{\pi\beta\, t}{2}\right)}{\beta\pi - 2\sin\frac{\pi\beta}{2}} \tag{65}$$

Figure 5 shows these cubic basis functions with $\beta = 1/2$.

Since S_i^3 is naturally reparametrized by $\phi(t)$, given the same four control points, the shapes of the curves whose basis functions are $S_i^3(t)$ and $Z_i^3(t)$, respectively are identical. Figure 6 shows the identical two curves defined with the same control points $(0,0)$, $(1,1)$, $(2,0)$, and $(3,1)$.

5 Quartic Curves

Since integration preserves reparametrization, we expect the shapes of the quartic curves are identical if those of their lower-degree curves are identical. Furthermore, if the shapes of the lower-degree curves are identical, those of the curves of any higer degree are identical. In this section we will confirm this fact for quartic curves.

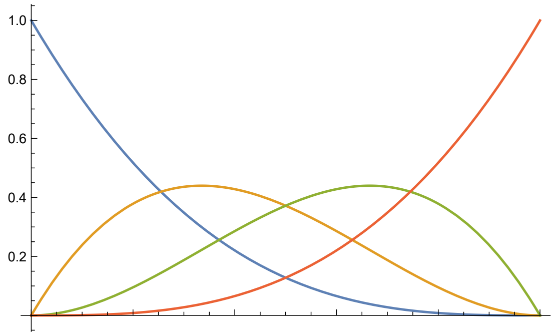

Figure 5: Basis functions of the up-degreed extended quadratic generalized trigonometric basis functions with $\beta = 1/2$.

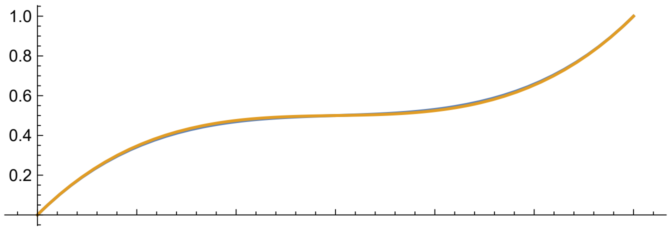

Figure 6: The identical two curves with $w = 1/\sqrt{2}$.

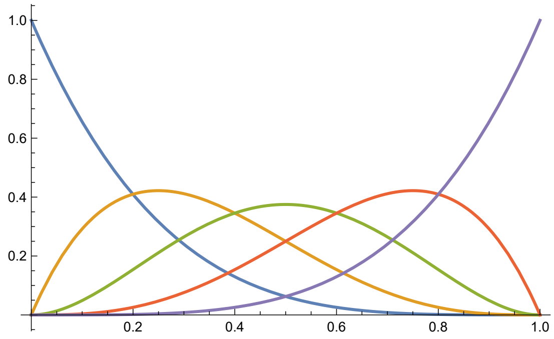

Figure 7: Quartic Rational Bézier Basis Functions with $w = \sqrt{\frac{1+\cosh\frac{1}{2}}{2}}$.

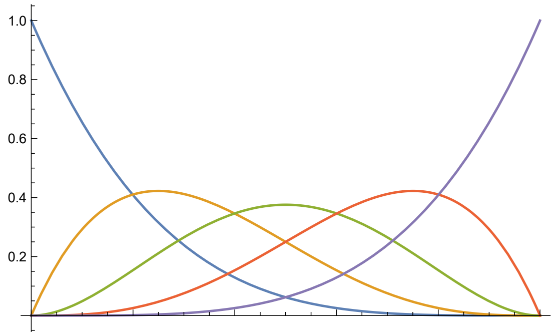

Figure 8: Quartic H-Bézier Basis Functions with $\alpha = 1/2$.

Appendices describe the basis functions of rational Bézier, H-Bézier and extended generalized trigonometoric curves of degree 4. Figures 7 and 8 show quartic rational Bézier basis functions with $w = \sqrt{\frac{1+\cosh\frac{1}{2}}{2}}$ and quartic H-Bézier basis functions with $\alpha = 1/2$ and figures 9 and 10 do quartic rational Bézier basis functions with $w = \cos\frac{\pi}{8}$ and quartic extended generalized trigonometric basis functions with $=1/2$, respectively.

Figure 11 shows the quartic rational Bézier curves with $\sqrt{\frac{1+\cosh\frac{1}{2}}{2}}$ and the quartic H-Bézier curves $a = 1/2$ are identical and figure 12 does the quartic rational Bézier curves with $w = \cos\frac{\pi}{8}$ and the quartic extended generalized trigonometric curves $=1/2$ are identical.

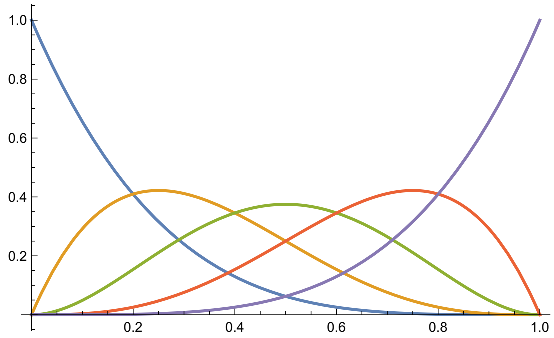

Figure 9: Quartic Rational Bézier Basis Functions with $w = \cos\frac{\pi}{8}$.

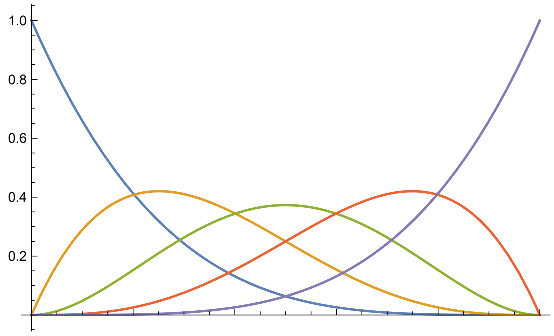

Figure 10: Quartic Extended Generalized Trigonometric Basis Functions with $b = 1/2$.

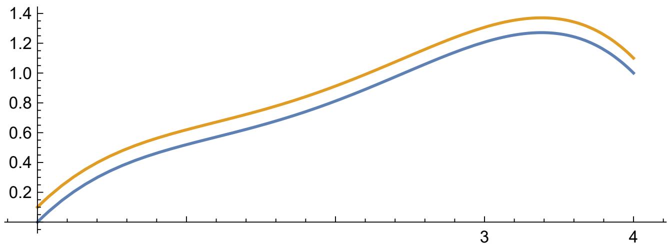

Figure 11: The identical two quartic curves with $\sqrt{\frac{1+\cosh\frac{1}{2}}{2}}$ for rational Bézier and $a = 1/2$ for H-Bézier curves. The second curve is translated in the vertical direction by 0.1.

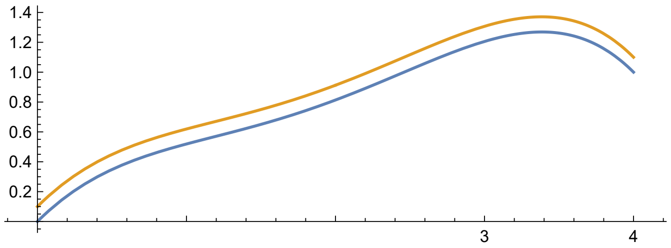

Figure 12: The identical two quartic curves with $w = \cos\frac{\pi}{8}$ for rational Bézier and $a = 1/2$ for extended generalized trigonometric curves. The second curve is translated in the vertical direction by 0.1.

6 C-Bézier Curve

The C-Bézier curve was proposed by Zhang [7] and it is defined by 4 control points using the following basis functions:

$$C_0^3(t) = \frac{(\alpha - t) - \sin(\alpha - t)}{\alpha - \sin\alpha} \tag{66}$$

$$C_1^3(t) = M[\frac{1 - \cos(\alpha - t)}{1 - \cos\alpha} - C_0^3(t)] \tag{67}$$

$$C_2^3(t) = M[\frac{1 - \cos t}{1 - \cos\alpha} - C_3^3(t)] \tag{68}$$

$$C_3^3(t) = \frac{t - \sin t}{\alpha - \sin\alpha} \tag{69}$$

where

$$M = \begin{cases} 1 & \text{if } \alpha = \pi \\ \frac{\sin\alpha(1-\cos\alpha)}{2\sin\alpha - \alpha - \alpha\cos\alpha} & \text{if } 0 \leq \alpha < \pi \end{cases} \tag{70}$$

where $0 < \alpha \leq \pi$.

In this definition the range of parameter t is $[0, \alpha]$ and we adopt its second form by reparametrizing as $s = t/\alpha$ as follows:

$$C_0^3(s) = \frac{\alpha(1 - s) - \sin\alpha(1 - s)}{\alpha - \sin\alpha} \tag{71}$$

$$C_1^3(s) = M[\frac{1 - \cos\alpha(1 - s)}{1 - \cos\alpha} - C_0^3(s)] \tag{72}$$

$$C_2^3(s) = M[\frac{1 - \cos\alpha s}{1 - \cos\alpha} - C_3^3(s)] \tag{73}$$

$$C_3^3(s) = \frac{\alpha t - \sin\alpha s}{\alpha - \sin\alpha} \tag{74}$$

We would like to reverse the recursive procedure described in Section 1 and obtain the lower version of these blending functions. By differentiating Eqs.(5), (6), and (7), with respect to s with Eq.(4), we obatin

$$\frac{dC_0^{n+1}(s)}{ds} = -\frac{C_0^n(s)}{D_0^n(1)} \tag{75}$$

$$\frac{dC_i^{n+1}(s)}{ds} = \frac{C_{i-1}^n(s)}{D_{i-1}^n(1)} - \frac{Z_i^n(s)}{D_i^n(1)} \quad (1 \le i \le n) \tag{76}$$

$$\frac{dC_{n+1}^{n+1}(s)}{ds} = \frac{C_i^n(s)}{D_n^n(1)} \tag{77}$$

where

$$D_i^n(s) = \int_0^s C_i^n(t)dt \quad (0 \le i \le n) \tag{78}$$

For the first the basis functions of the C-Bézier curve, we obtain from Eq.(75),

$$C_0^2(t) = -\frac{dC_0^3(s)}{ds} D_0^2(1) \tag{79}$$

We assume when $s = 0$, $C_0^2(0) = 1$. Since

$$\frac{dC_0^3(s)}{ds} = \frac{\alpha(\cos\alpha(1-s)-1)}{\alpha - \sin\alpha}, \tag{80}$$

then

$$\frac{dC_0^3(0)}{ds} = \frac{\alpha(\cos\alpha - 1)}{\alpha - \sin\alpha}, \tag{81}$$

Hence

$$D_0^2(1) = -\frac{1}{\frac{dC_0^3(0)}{ds}} = -\frac{\alpha - \sin\alpha}{\alpha(\cos\alpha - 1)} \tag{82}$$

Therefore

$$C_0^2(t) = \frac{\alpha(\cos\alpha(1-s)-1)}{\alpha - \sin\alpha} \frac{\alpha - \sin\alpha}{\alpha(\cos\alpha - 1)} = \frac{1-\cos\alpha(1-s)}{1-\cos\alpha} \tag{83}$$

We assume that $C_2^2(s) = C_0^2(s)$ and $C_1^2(s) = 1 - C_0^2(s) - C_0^2(s)$ and

$$C_1^2(s) = \frac{\cos(\alpha s) + \cos\alpha(1-s) - \cos\alpha - 1}{1-\cos\alpha} \tag{84}$$

$$C_2^2(s) = \frac{1-\cos\alpha s}{1-\cos\alpha} \tag{85}$$

Note that

$$\frac{C_1^2(s)^2}{C_0^2(s)C_2^2(s)} = 2(1+\cos\alpha) \tag{86}$$

Therefore its equivalent weight $w_c = \sqrt{\frac{1+\cos\alpha}{2}}$. Since $0 < \alpha \le \pi$, $0 < w_c \le 1$.

Similarly we obtain

$$C_0^1(s) = \frac{\sin\alpha(1-s)}{\sin\alpha} \tag{87}$$

$$C_1^1(s) = \frac{\sin\alpha s}{\sin\alpha} \tag{88}$$

When $\alpha = \pi\beta/2$, Eqs.(52) and (53) become the same.

Let's review the definition of the second form of the C-Bézier curve and we can rewrite them as

$$C_1^3(s) = M[C_0^2(s) - C_0^3(s)] \tag{89}$$

$$C_2^3(s) = M[C_2^2(s) - C_3^3(s)] \tag{90}$$

M is automatically determined by the partition of unity of the basis functions.

In conclusion, the basis functions of the C-Bézier curve consist of the mixture of the quadratic and cubic basis functions of the extended generalized trigonometric functions.

7 CONCLUSIONS

In this paper we have applied the shape uniquness theorem to the H-Bézier curve [3], whose blending functions are defined by recursively using integral forms. We have shown the equivalence of quadratic H-Bézier and quadratic rational Bézier curves and that of quadratic extended generalized trigonometric ones. Also we have shown that the reparametrization is kept by the integration and if the original curves are equivalent, the recursive procedure guarantees the equivalence of the up-degreed curves. Furthermore we have derived new basis functions of degrees 3 and 4 based on the construction of H-Bézier basis functions.

ACKNOWLEDGEMENTS

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A Quartic rational Bézier basis functions

The basis functions in Appendices A, B and C are “symmetrical”, that means

$$Z_3^4(t) = Z_{r1}^4(1-t) \tag{91}$$

$$Z_4^4(t) = Z_{r0}^4(1-t) \tag{92}$$

Hence, we write only $Z_0^4(t)$, $Z_1^4(t)$ and $Z_2^4(t)$.

$$Z_{r0}^4(t) = \frac{f_{r0}(t)}{g_{r0}(t)} \tag{93}$$

$$Z_{r1}^4(t) = \frac{f_{r1}(t)}{g_{r1}(t)} \tag{94}$$

$$Z_{r2}^4(t) = \frac{f_{r2}(t)}{g_{r2}(t)} \tag{95}$$

$$\tag{96}$$

where

$$f_{r0}(t) = 1 - (\frac{f_{nr0}(t)}{f_{dr0}(t)} - \tan^{-1}\left(\frac{\sqrt{w-1}}{\sqrt{-w-1}}\right)) \tag{97}$$

$$g_{r0}(t) = -\sqrt{-w-1}\sqrt{w-1} - 2\tan^{-1}\left(\frac{\sqrt{w-1}}{\sqrt{-w-1}}\right) \tag{98}$$

$$\begin{aligned} f_{r1}(t) = {} & \frac{1}{2\sqrt{w-1}}(\frac{\sqrt{-w-1}(2t^2(w-1) + (-2tw+2t-1)\log(2t(t(-w)+t+w-1)+1)) + h_{r1}(t)}{\sqrt{-w-1}\sqrt{w-1} + 2\tan^{-1}(\frac{\sqrt{w-1}}{\sqrt{-w-1}})} \\ & -2\sqrt{-w-1}(\tan^{-1}\left(\frac{\sqrt{w-1}}{\sqrt{-w-1}}\right)\left((w-1)(2t(t(-w)+t+w+1) + (1-2t)\log(2t(t(-w)+t+w-1)+1)+2) - 2\sqrt{-w-1}\sqrt{w-1}\tan^{-1}\left(\frac{(2t-1)\sqrt{w-1}}{\sqrt{-w-1}}\right)\right) \\ & +(w-1)\left(2(t(w-1)+1)\tan^{-1}\left(\frac{(2t-1)\sqrt{w-1}}{\sqrt{-w-1}}\right) - t\sqrt{-w-1}\sqrt{w-1}(\log(2t(t(-w)+t+w-1)+1)-2)\right) - 2\sqrt{-w-1}\sqrt{w-1}\tan^{-1}\left(\frac{\sqrt{w-1}}{\sqrt{-w-1}}\right)^2) \end{aligned} \tag{99}$$

$$g_{r1}t = -2w^2 + 4(w+1)\tan^{-1}\left(\frac{\sqrt{w-1}}{\sqrt{-w-1}}\right)^2 + 2(w+3)\sqrt{-w-1}\sqrt{w-1}\tan^{-1}\left(\frac{\sqrt{w-1}}{\sqrt{-w-1}}\right) + 2\Bigg) \tag{100}$$

$$f_{r2}(t) = -\left((w-1)\left((w+1)\log(2t(t(-w)+t+w-1)+1) - 2\sqrt{-w-1}\sqrt{w-1}\left((2(t-1)t+1)\tan^{-1}\left(\frac{\sqrt{w-1}}{\sqrt{-w-1}}\right) + (1-2t)\tan^{-1}\left(\frac{(2t-1)\sqrt{w-1}}{\sqrt{-w-1}}\right)\right)\right)\right) \tag{101}$$

$$g_{r2}(t) = 2\left(w^2 - \tan^{-1}\left(\frac{\sqrt{w-1}}{\sqrt{-w-1}}\right)\left((w+3)\sqrt{-w-1}\sqrt{w-1} + 2(w+1)\tan^{-1}\left(\frac{\sqrt{w-1}}{\sqrt{-w-1}}\right)\right) - 1\right) \tag{102}$$

where

$$f_{nr0}(t) = -2t^2\sqrt{-w-1}w + 2t^2\sqrt{-w-1} - 2t\sqrt{-w-1}\log\left(-2t^2(w-1)+2t(w-1)+1\right)$$
$$+ 2tw\sqrt{-w-1}\log\left(-2t^2(w-1)+2t(w-1)+1\right) + \sqrt{-w-1}\log\left(-2t^2(w-1)+2t(w-1)+1\right)$$
$$+ 4tw\sqrt{w-1}\tan^{-1}\left(\frac{\sqrt{w-1}}{\sqrt{-w-1}}\right) + 4tw\sqrt{w-1}\tan^{-1}\left(\frac{(1-2t)\sqrt{w-1}}{\sqrt{-w-1}}\right)$$
$$- 2\sqrt{w-1}\tan^{-1}\left(\frac{(2t-1)\sqrt{w-1}}{\sqrt{-w-1}}\right) \quad (103)$$

$$f_{dr0}(t) = 2\sqrt{w-1} \quad (104)$$

$$h_{r1}(t) = 2\sqrt{w-1}(1-2tw)\tan^{-1}\left(\frac{\sqrt{w-1}}{\sqrt{-w-1}}\right) + 2\sqrt{w-1}(2tw+1)\tan^{-1}\left(\frac{(2t-1)\sqrt{w-1}}{\sqrt{-w-1}}\right) \quad (105)$$

B Quartic H-Bézier basis functions

$$Z^4_{h0}(t) = \frac{a^2(t-1)^2 - 2\cosh(a-at)+2}{a^2 - 2\cosh(a)+2} \quad (106)$$

$$Z^4_{h1}(t) = \frac{f^4_{h1}}{g^4_{h1}} \quad (107)$$

$$Z^4_{h2}(t) = \frac{a(a(t-1)t + \sinh(at) + \sinh(a-at) - a(t-1)t\cosh(a) - \sinh(a))}{\left(a\cosh\left(\frac{a}{2}\right) - 2\sinh\left(\frac{a}{2}\right)\right)^2} \quad (108)$$

where

$$f^4_{h1}(t) = (\sinh(a)-a)(at\left(a^2(t-1)+2\right) + 2\sinh(at) + 2\sinh(a-at)$$
$$+ a(a\left(\sinh(a-at) - (t-1)^2\sinh(a)\right) - 2(t-1)\cosh(a) - 2\cosh(a-at)) - 2\sinh(a)) \quad (109)$$

$$g^4_{h1}(t) = \left(a^2 - 2\cosh(a)+2\right)\left(a\cosh\left(\frac{a}{2}\right) - 2\sinh\left(\frac{a}{2}\right)\right)^2 \quad (110)$$

C Quartic extended generalized trigonometric basis functions

$$Z^4_{t0}(t) = \frac{\pi^2\beta^2(t-1)^2 + 8\cos\left(\frac{1}{2}\pi\beta(t-1)\right) - 8}{\pi^2\beta^2 + 8\cos\left(\frac{\pi\beta}{2}\right) - 8} \quad (111)$$

$$Z^4_{t1}(t) = \frac{\pi^2\left(-\beta^2\right)(t-2)t - 8\cos\left(\frac{1}{2}\pi\beta(t-1)\right) + 8\cos\left(\frac{\pi\beta}{2}\right)}{\pi^2\beta^2 + 8\cos\left(\frac{\pi\beta}{2}\right) - 8}$$
$$- \frac{\pi\beta(\pi\beta - 2\sin(\frac{\pi\beta}{2}))(f_{zt1}(t) - g_{zt1}(t))}{-\pi\beta\sin\left(\frac{\pi\beta}{2}\right) - 4\cos\left(\frac{\pi\beta}{2}\right) + 4} \quad (112)$$

$$Z^4_{t2}(t) = \frac{\pi\beta\tan\left(\frac{\pi\beta}{4}\right)\left(\pi\beta(t-1)t\sin\left(\frac{\pi\beta}{2}\right) + 2\left(\cos\left(\frac{\pi\beta t}{2}\right) + \cos\left(\frac{1}{2}\pi\beta(t-1)\right) - 1\right) - 2\cos\left(\frac{\pi\beta}{2}\right)\right)}{\left(\pi\beta\cos\left(\frac{\pi\beta}{4}\right) - 4\sin\left(\frac{\pi\beta}{4}\right)\right)^2} \quad (113)$$

$$(114)$$

where

$$f_{zt1}(t) = \frac{\frac{1}{2}\pi\beta t^2 - 2t\sin\left(\frac{\pi\beta}{2}\right) + \frac{4\left(\cos\left(\frac{1}{2}\pi\beta(t-1)\right)-\cos\left(\frac{\pi\beta}{2}\right)\right)}{\pi\beta}}{\pi\beta - 2\sin\left(\frac{\pi\beta}{2}\right)} \tag{115}$$

$$g_{zt1}(t) = \frac{\csc\left(\frac{\pi\beta}{4}\right)\sec\left(\frac{\pi\beta}{4}\right)\left(\pi^2\beta^2t^2\left(\cos\left(\frac{\pi\beta}{2}\right)+1\right) - 4\pi\beta t\sin\left(\frac{\pi\beta}{2}\right) + 8\left(\cos\left(\frac{\pi\beta t}{2}\right) + \cos\left(\frac{1}{2}\pi\beta(t-1)\right) - \cos\left(\frac{\pi\beta}{2}\right) - 1\right)\right)}{4\pi\beta\left(\pi\beta\cot\left(\frac{\pi\beta}{4}\right) - 4\right)} \tag{116}$$