Least Squares Fitting on a Segment of Ellipse and Its Application on Road Curvature Estimation

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Abstract. In least squares fitting, the best fitting curve to a given set of points can be obtained by minimizing the sum of the square errors of the points from the curve, hence enabling a solution for the parameters and the production of an ellipse. In this paper, the least squares fitting of a curve is computed by using parameterization of an ellipse. The paper focuses on datasets that form a segment of ellipse rather than forming a full ellipse. Different z-values from parametric equations that ranges between 0 to 2π radians will give different curve fittings to the given data. However, a simple algorithm that uses a numerical optimization technique called simulated annealing is proposed to obtain the optimum z-values that yield the best curve fitting for any given data in a segment of an ellipse. Subsequently, road curvature estimation can be computed using this algorithm which can potentially be utilized in real-life applications particularly within the engineering domain.

Keywords: Least squares fitting, curve fitting, parametric equation, ellipse, minimization
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1 INTRODUCTION

The fitting of geometric features such as circle and ellipse to given points is applicable in various fields [21]. Ellipses for example, may represent many real-world situations such as orbits of the planets, satellites, and comets. These conic sections are also relevant in science, astronomy, and variety of engineering applications [8]. In real-world practice, least squares fitting is used in various manufacturing industries and software applications [23]. Indirectly, it is also significant in pattern recognition [5] and Computer-Aided Design (CAD).

A circle, an ellipse, a parabola or a hyperbola are curves that are referred to as conic sections. These four conic sections are derived from intersecting a cone with a plane in four distinct and non-trivial relative positions [14]. Meanwhile, Aguirre-Ramos et al. [1] define a conic section as the result of intersecting a cutting plane with a double right circular cone, which produces a regular 2D curve in the form of a circle, a parabola, an ellipse, or a hyperbola. By incorporating method such as changing the inclination angle of the plane, the resulting curve can be obtained.

Elliptical arcs are preferred over general conic arcs as ellipses are frequently encountered shapes found as a component in various objects and have been proven to be beneficial in fields such as Computer-Aided Design (CAD), computer graphics and computer vision [18]. For instance, we are interested to fit data that forms an arc or a segment of an ellipse. Hence, in this paper we apply ellipse fitting rather than other curve fitting. In addition, ellipse ranks among the most prevalent geometric shapes found in the real world [22].

For geometric features, there are two main classifications of least squares fitting problems, namely algebraic and geometric fitting whereby they are differentiated by the definition of error distances [3]. For algebraic fitting, it uses implicit form of the conic section [9]. Deviations of the implicit form from the expected value are described as the error distances. Ahn et al. [3] describes the error distances for geometric fitting by computing the orthogonal or shortest distances from the given points to the fitted curve of the geometric features. The best curve fitting is obtained by trying to minimize the orthogonal distance to the curve [4].

In this paper, we are interested to fit parametric curves in the least squares sense. The best fitting curve to a given set of points is obtained by minimizing the sum of the square errors of the points from the curve [2]. This mathematical procedure is called least squares fitting. Gander et al. [11], Watson [23], and Pilu et al. [17] apply least squares fitting based on data that forms entire circles and ellipses but here, we only discuss a part of an ellipse or a circle that represents an arc [6] or a curve. While the parameter z for a full ellipse ranges between 0 to 2π radians, the challenge with data that form partial ellipse is to determine where it lies in the parameter z. Firstly, we will determine the parameter z that best fits the data points presumed to be part of an ellipse. We will then demonstrate how this parameter can be utilized to estimate road curvature and compare it with the experimental approach employed by engineers.

A simple algorithm that uses a numerical optimization technique is introduced to obtain the best curve fitting for any given set of data, specifically data that forms a segment of an ellipse. This technique is known as simulated annealing. Structural optimization problems use simulated annealing extensively due to its inherent simplicity and capacity to discover the global optimum even when dealing with numerous design variables [13]. There are extensive usage of simulated annealing in real-life applications in which the primary benefit of it lies in its simplicity [7]. It is known that simulated annealing demonstrates efficiency, ease of implementation, and theoretically reliable but there are also disadvantages such as a slow convergence rate [12].

The general parametric conic arcs introduced by [10] outline the process of creating conic blending arcs by utilizing a unified rational parametric representation that merges the distinct cases of blending parallel and non-parallel edges based on given constraints such that it requires the arc to maintain a specified distance from a line, point, or a circle. Otherwise, intersect a circle or line at a predetermined angle. In this paper, instead of interpolating points, we used least squares fitting to approximate points for any given data.

2 LEAST SQUARES FITTING ON A SEGMENT OF AN ELLIPSE

2.1 Fitting An Ellipse In Parametric Form

Generally, in order to fit an ellipse in parametric form, we follow Späth [19] and consider the equations:

$$\begin{aligned} x(z) &= a + p\ cos(z) \\ y(z) &= b + q\ sin(z) \end{aligned} \tag{1}$$

where (a, b) is the center of the ellipse, p is the radius along the x-axis, q is the radius along the y-axis and parameter z lies between 0 to 2π radians. The function to be minimized is:

$$S(a,b) = \sum_{i=0}^{n} (x_i - a - p\ \ cos(z_i))^2 + (y_i - b - q\ \ sin(z_i))^2, \tag{2}$$

where z_i is a parametrized value that lies between 0 to 2π radians and (x_i, y_i) are data points.

Späth [19] utilizes (1) to fit an ellipse based on data forming a complete elliptical shape. However, we consider the case where collected data are from a segment and not the entire ellipse. The issue is dealt with in [4]; however, we use a parameterization approach in minimization as described in the next section.

In this case, we aim to minimize (2) where z_i is a parameterized value that lies between θ_1 and θ_2. Parameter z_i should cover a certain part of an ellipse. For instance, if the data is half of an ellipse that forms the upper half of ellipse, z_i should cover from 0 to π radians. If let's say the data forms the bottom half of the ellipse, then z_i can be from π to 2π radians. The range can be determined through observation; nonetheless, we will select the optimal values for θ_1 and θ_2 by optimizing equation (2).

The values of a, b, p and q can be solved by differentiating (2) with respect to each parameter and equate it to 0:

$$\frac{\delta S}{\delta a} = 0 \ , \frac{\delta S}{\delta b} = 0 \ , \frac{\delta S}{\delta p} = 0 \ , \frac{\delta S}{\delta q} = 0. \tag{3}$$

The presence of parameter z_i and the uncertainty regarding its interval render the problem difficult to solve. Therefore, we will determine the values of a, b, p, and q using simulated annealing, a method that will be further elaborated in Section 2.2

2.2 Minimizing The Error Distances

To establish the minimization process using simulated annealing, we start by discussing about parameter z in the ellipse function. The parameter z_i can be computed by:

$$z_i = \theta_1 + (i-1)h, \tag{4}$$

where $i = 1, 2, ..., n$ and θ_1 denotes the start of the interval while h is defined as the step size for parameter z_i and it is assigned arbitrarily.

For instance, if $h = 0.1$, then $z = \{\theta_1, \theta_1 + 0.1, \theta_1 + 0.2, \theta_1 + 0.3, ...\}$. By choosing any value from θ_1 to θ_2 from this range, where $\theta_2 = \theta_1 + (n-1)h$, we can see the pattern of the error distance to be either increasing or decreasing. To minimize (2), we employ a numerical approach to evaluate the value of $S(a,b)$. We aim to find the values of θ_1 and h that will minimize the function.

The error distance can be obtained by using:

$$d = \sum_{i=0}^{n} |(X_i, Y_i) - (x_i, y_i)|, \tag{5}$$

where (X_i, Y_i) are the points on the estimated curve and (x_i, y_i) are the original data points. Hence the minimum value of (5) is the solution to the minimization problem.

We perform the minimization by using a numerical optimization technique called simulated annealing which is available as a built-in function in Mathematica. The purpose is to find the optimum values of z, and parameters a, b, p, q, θ_1 and h in order to obtain the best curve fitting of an ellipse. The next part of the algorithm is to input data i.e. number of observations, n, coordinates (x_1, y_1), (x_2, y_2),..., (x_n, y_n), v_1, and v_2. Then, minimizing (2) using simulated annealing subject to constraint $0 \leq \theta_1 \leq 2\pi$ and $v_1 \leq h \leq v_2$, whereas v_1 and v_2 are the minimum and maximum step sizes respectively. For our experiment, we let v_1 and v_2 range from 0.1 to 0.5 where $v_1 < v_2$. As a result, the output obtained are the parameters a, b, p, and q for the equation of ellipse, the minimum error distance, and the values of θ_1 and h. For further reference, the coding for this algorithm is included in Fig. 1.

Input Data

The data in *x* and *y* coordinates as shown in Table 1.

```
In[6]:= data = {{5.52445, 3.8379}, {4.36629, 4.12056}, {3.10566, 4.11217}, {1.85366, 3.99601},
   {0.703304, 3.6778}, {-0.324662, 3.42801}, {-1.10022, 2.99983}, {-1.74638, 2.67906},
   {-2.07959, 2.18775}};
n = 9;
x = Table[data[[j, 1]], {j, 1, n}];
y = Table[data[[j, 2]], {j, 1, n}];
z[i_, theta1_, zh_] := theta1 + (i - 1) * zh;
solution =
 NMinimize[
  {Sum[(x[[i]] - a - p Cos[z[i, theta1, zh]])^2, {i, 1, n}] + Sum[(y[[i]] - b - q Sin[z[i, theta1, zh]])^2, {i, 1, n}],
   0 ≤ theta1 ≤ 2 Pi, 0.1 ≤ zh ≤ 0.3}, {a, b, p, q, theta1, zh}, Method → "SimulatedAnnealing"]
```

Output Data

```
Out[10]= {0.0374444, {a → 3.56603, b → 1.76578, p → 5.72653, q → 2.29323, theta1 → 1.21592, zh → 0.21939}}
```

Figure 1: The Mathematica coding used for minimization.

3 DATA ANALYSIS

This section introduces least squares fitting of a curve based on the parameterization of an ellipse. Spath [19] proposed that the fitting of an ellipse follows Equation (1). Instead of generalisation, we choose a set of data that forms a segment of an ellipse to demonstrate the difference between our approach and the literature. The data set is generated arbitrarily from an ellipse with the addition of some noise. Table 1 shows a sample data for further analysis. The data are taken in x and y coordinates, as depicted in the original data set shown in Fig. 2a. This section also discusses minimum value of the error distances which is the solution to the minimization problem in (5). Both are pertinent as the solution to the minimization problem will give the best curve fitting for the given data.

To obtain the best curve fitting for the given data from Table 1, we perform the proposed algorithm according to the specification outlined in Section 2.2. A critical step in finding minimization using simulated annealing is to define an appropriate interval of h which are the values of v_1 and v_2. If one defines v_1 and v_2 as too small, for example $v_1 = 0$ and $v_2 = 0.05$, then the parameter z will not be able to cover the whole data points. For this example, several intervals are used for h within 0.1 and 0.5. By defining $v_1 = 0.1$ and $v_2 = 0.3$, we obtained the value of $\theta_1 = 1.21592$ and $h = 0.21939$ with a minimum error distance of 0.037444. The best curve fitting is shown in Fig. 2b which yields $a = 3.56603$, $b = 1.76578$, $p = 5.72653$, and $q = 2.29323$ forming a complete ellipse as shown in Fig. 2c that fits the points on a segment nicely.

Hence, we observe that $\theta_1 = 1.21592$, which is approximately equal to 69.667 degrees, representing the orientation between the origin and the initial point. Consequently, $\theta_2 = 1.21592 + 8(0.21939) = 2.97104$. This is equivalent to 170.23 degrees, which is approximately the angle between the origin and the final point.

For the regularity of parameterization, our current setting is optimal based on observations and verification with real-life data. We note that the formulation defined in this approach is uniform because the data is self-selected, resulting in behaviour that is close to uniform, thus yielding good results. In cases where the data is non-uniform, we need to extend our approach, as the parameterization must be adjusted to accommodate

Points	Coordinates
1	(5.52445, 3.83790)
2	(4.36629, 4.12056)
3	(3.10566, 4.11217)
4	(1.85366, 3.99601)
5	(0.70330, 3.67780)
6	(-0.32466, 3.42801)
7	(-1.10022, 2.99983)
8	(-1.74638, 2.67906)
9	(-2.07959, 2.18775)

Table 1: The data in x and y coordinates.

non-uniform data.

4 THE APPLICATION OF ELLIPSE FITTING ON ROAD CURVATURE ESTIMATION

After the minimization procedure, during which we obtained the best-fitting curve for data forming a segment of an ellipse, we now aim to apply the proposed algorithm to fit data from small segments of roads, particularly those with curvy shapes. Few points will be taken along the desired segment of a road. Hence, fitting an ellipse on the segment of a road will allow us to calculate the radius of curvature for each point precisely based on its coordinate on the road.

For the radius of curvature, we compare our approach to Luo et al. [15]. In their paper, radius of curvature was calculated at 9 different test sites, chosen from highway ramps and field measurement was used to conduct the validation tests. Besides, this paper uses roadway centerline to measure the radius of curvature and curve length.

In the following subsection, we will determine the road coordinates by referencing the selected test sites stated in [15]. Ellipses will be constructed, and the equation obtained will be used to estimate the radius of curvature. The radius of curvature can be calculated by using (6):

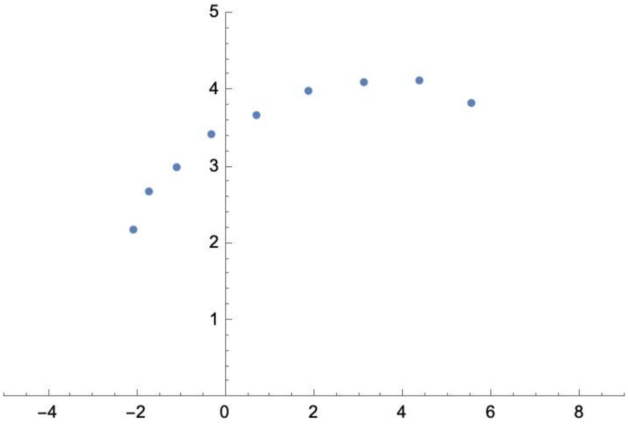
$$R = \frac{[(x')^2 + (y')^2]^{\frac{3}{2}}}{|x'y" - y'x"|} \tag{6}$$

where $x(z)$ and $y(z)$ are from Equation 1.

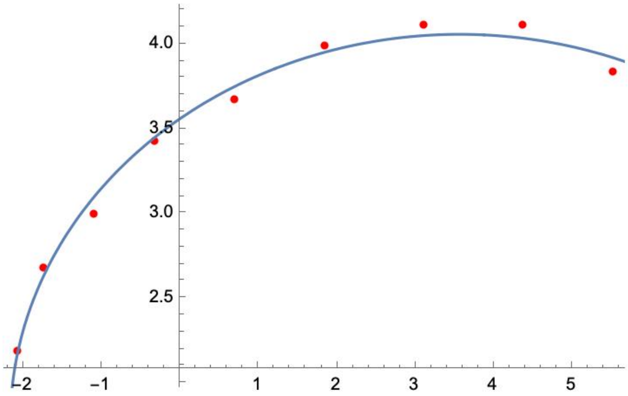
4.1 Test Site 3

Test Site 3 as shown in Fig. 3 is located in Interstate 35 (I-35) in Kansas, United States that begins at 39°02'20.43" N, 94°40'26.76" W and ends at 39°02'29.26" N, 94°40'22.67" W with the length of 324 m. The radius of curvature obtained from field measurement is 104.15 m [7]. Coordinates of 9 points along Test Site 3 from Google Maps were chosen and presented in Table 2 and the best curve from minimization procedure is fitted as shown on the left side of Fig. 4. Meanwhile, the right side of Fig. 4 displayed the fitting of a full ellipse.

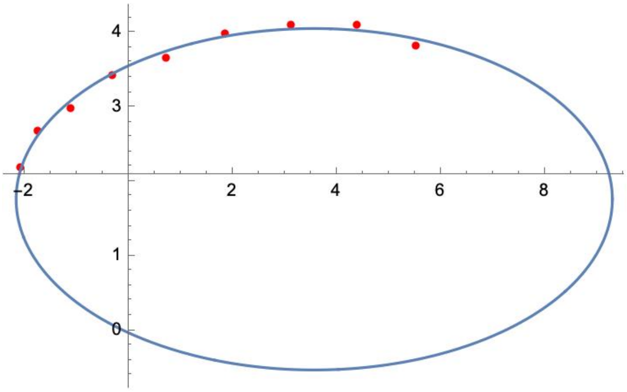
We can observe that the centre of the ellipse segment is in between point 4 and point 5 and the radius of curvature in between those points lies between 123.686 m and 94.6137 m. The radius of curvature obtained



(a) The plotted data of 9 coordinates.



(b) The best curve fitting of the 9 coordinates.



(c) Fitting of a full ellipse of the 9 coordinates.

Figure 2: A segment of an ellipse using 9 coordinates.

from our proposed algorithm is nearly equal to the radius of curvature found by Luo et al. [15] which is 104.15 m. We do not provide an exact comparison as we are uncertain of which specific point is referenced in [15].

Points	Coordinates	Radius of curvature (m)
1	(39.039408, -94.672916)	293.312
2	(39.039537, -94.672637)	213.213
3	(39.039712, -94.672455)	157.344
4	(39.039871, -94.672347)	123.686
5	(39.040104, -94.672268)	94.6137
6	(39.040312, -94.672229)	85.0239
7	(39.040496, -94.672251)	89.7698
8	(39.040704, -94.672315)	110.282
9	(39.040944, -94.672433)	153.020

Table 2: The coordinates and radius of curvature for 9 points taken along Test Site 3.

Figure 3: Location of Test Site 3 on Google Maps.

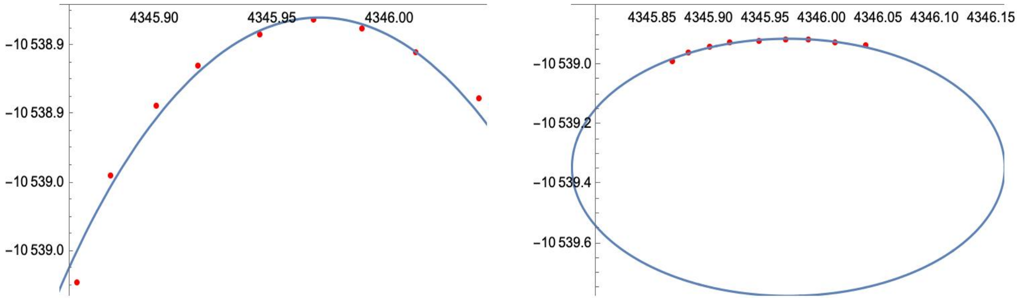


Figure 4: Left: Best curve fitting along Test Site 3; Right: Fitting of a full ellipse on Test Site 3.

4.2 Test Site 5

Another example is demonstrated on Test Site 5 as shown in Fig. 5 which is located in Interstate 69 (I-69) in Michigan, United States that begins at 42°59'7.26" N, 83°43'55.37" W and ends at 42°59'8.03" N, 83°44'3.70" W with the length of 558 m. The radius of curvature obtained from field measurement is 70.02 m [7]. Coordinates of 11 points along Test Site 5 from Google Maps were chosen and presented in Table 3 and the best curve from minimization procedure is fitted as shown on the left side of Fig. 6. Meanwhile, the right side of Fig. 6 displayed the fitting of a full ellipse. Here, more points are taken along Test Site 5 to further prove the significance of the findings.

Points	Coordinates	Radius of curvature (m)
1	(42.986070, -83.735240)	646.703
2	(42.986047, -83.735622)	419.772
3	(42.985966, -83.735905)	266.901
4	(42.985827, -83.736163)	145.825
5	(42.985630, -83.736310)	76.2067
6	(42.985409, -83.736378)	42.8404
7	(42.985167, -83.736384)	50.4153
8	(42.984951, -83.736249)	101.878
9	(42.984772, -83.736009)	209.18
10	(42.984650, -83.735730)	357.332
11	(42.984610, -83.735478)	500.231

Table 3: The coordinates and radius of curvature for 11 points taken along Test Site 5.

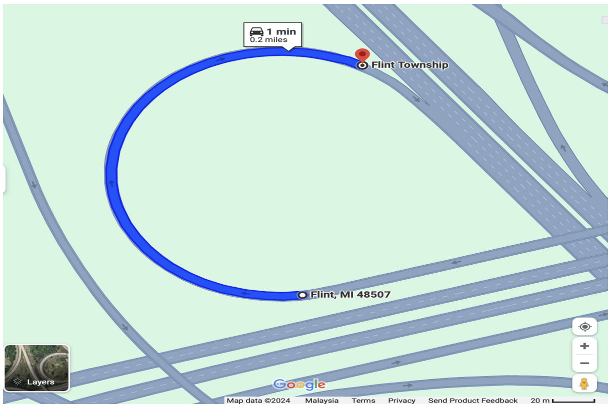


Figure 5: Location of Test Site 5 on Google Maps.

We can observe that the centre of the ellipse segment is in between point 5 and point 6 and the radius of curvature in between those points lies between 76.2067 m and 42.8404 m. The radius of curvature obtained

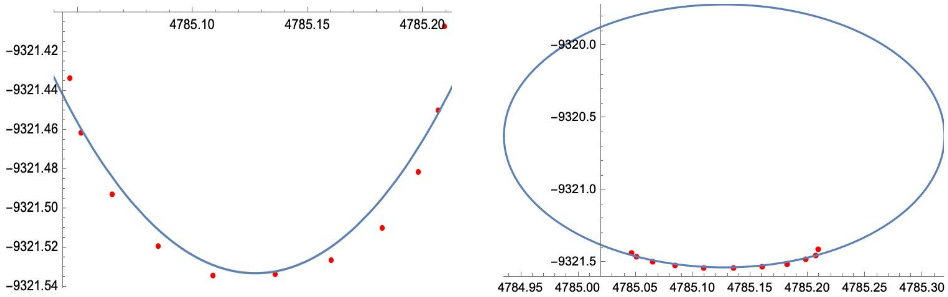


Figure 6: Left: Best curve fitting along Test Site 5; Right: Fitting of a full ellipse on Test Site 5.

from our proposed algorithm is nearly equal to the radius of curvature found by Luo et al. [15] which is 70.02 m. Hence, the results demonstrate the significance of our approach in obtaining the radius of curvature.

The method used in our paper shows a high similarity to the field measurement in which radius of curvature obtained from the least squares fitting on a segment of an ellipse is found to be approximately equal to the radius of curvature obtained by Luo et al. [15].

5 CONCLUSIONS

In this paper, least squares fitting is applied to obtain the best curve fitting to the given data that form a segment of ellipse by minimizing the sum of square errors using simulated annealing. It can be observed that the solution to the minimization problem approximates the data closely by the ellipse. In addition, we fit the data of a small segment of a road to obtain its curvature at any specific point on the road. For perspective, this can be extended in future research for travel time prediction in [20] or for the purpose of road safety in [16]. The positive aspect of our approach lies in its cost efficiency as we rely on the readily available GPS data. Generally, if a set of a parametric data is assumed to behave in ellipse shape, we should be able to perform least squares fitting using the proposed algorithm. A few segment of roads have been tested by using this approach and the results demonstrated were proven reliable by the proposed algorithm in obtaining the radius of curvature. In the future, it would be advantageous for us to incorporate the non-uniform parameterization and consider the case where noise is added to the data. The appropriateness of applying uniform and non-uniform parameterization in the presence of noise will be observed. This would strengthen our approach from another point of view.

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